

ASYMPTOTIC MULTIPLICITIES OF GRADED FAMILIES OF IDEALS AND LINEAR SERIES

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ABSTRACT. We find simple necessary and sufficient conditions on a local ring R of dimension d for the limit

$$\lim_{i \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d}$$

to exist whenever $\{I_n\}$ is a graded family of m_R -primary ideals, and give a number of applications. We also give simple necessary and sufficient conditions on projective schemes over a field k for asymptotic limits of the growth of all graded linear series of a fixed Kodaira-Iitaka dimension to exist.

1. INTRODUCTION

1.1. Limits of graded families of ideals. In this paper we prove the following theorem about graded families of m_R -primary ideals.

Theorem 1.1. *(Theorem 5.5) Suppose that R is a (Noetherian) local ring of dimension d , and $N(\hat{R})$ is the nilradical of the m_R -adic completion \hat{R} of R . Then the limit*

$$(1) \quad \lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d}$$

exists for any graded family $\{I_n\}$ of m_R -primary ideals, if and only if $\dim N(\hat{R}) < d$.

A graded family of ideals $\{I_n\}$ in R is a family of ideals indexed by the natural numbers such that $I_0 = R$ and $I_m I_n \subset I_{m+n}$ for all m, n .

The nilradical $N(R)$ of a d -dimensional ring R is

$$N(R) = \{x \in R \mid x^n = 0 \text{ for some positive integer } n\}.$$

Recall that $\dim N(R) = -1$ if $N(R) = 0$ and if $N(R) \neq 0$, then

$$\dim N(R) = \dim R / \text{ann}(N(R)),$$

so that $\dim N(R) = d$ if and only if there exists a minimal prime P of R such that $\dim R/P = d$ and R_P is not reduced.

If R is excellent, then $N(\hat{R}) = N(R)\hat{R}$, and the theorem is true with the condition $\dim N(\hat{R}) < d$ replaced with $\dim N(R) < d$. However, there exist Noetherian local domains R (so that $N(R) = 0$) such that $\dim N(\hat{R}) = \dim R$ (Nagata (E3.2) [32]).

It is not difficult to construct examples of graded families of m_R -primary ideals in a regular local ring such that the above limit is irrational.

The problem of existence of such limits (1) has been considered by Ein, Lazarsfeld and Smith [15] and Mustařă [31]. Lazarsfeld and Mustařă [28] showed that the limit exists for all graded families of m_R -primary ideals in R if R is a domain which is essentially of finite type over an algebraically closed field k with $R/m_R = k$. All of these assumptions are necessary in their proof. Their proof is by reducing the problem to one on graded linear

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series on a projective variety, and then using a method introduced by Okounkov [33] to reduce the problem to one of counting points in an integral semigroup.

In our paper [6], we prove that such limits exist for graded families of m_R -primary ideals, with the restriction that R is an analytically unramified ($N = 0$) equicharacteristic local ring with perfect residue field (Theorem 5.8 [6]). In this paper we extend this result (in Theorem 4.7) to prove that the limit (1) exists for all graded families of m_R -primary ideals in a local ring R satisfying the assumptions of Theorem 1.1, establishing sufficiency in Theorem 1.1. Our proof begins with the cone method discussed above.

In Example 5.3, we give an example of a graded family of m_R -primary ideals in a nonreduced local ring R for which the above limit does not exist. Hailong Dao and Ilya Smirnov have shown that such examples are universal, so that if the nilradical of R has dimension d , then there exists a graded family of m_R -primary ideals such that the limit (1) does not exist (Theorem 5.4 of this paper). Since a graded family of m_R -primary ideals on the completion of a ring lifts to the ring, necessity in Theorem 1.1 follows from this result.

In Section 6 of this paper, we give some applications of this result and the method used in proving it, which generalize some of the applications in [6]. We extend the theorems to remove the requirement that the local ring be equicharacteristic with perfect residue field, to hold on arbitrary analytically unramified local rings.

We prove some volume = multiplicity formulas for graded families of m_R -primary ideals in analytically unramified local rings in Theorems 6.5 - 6.9. Theorem 6.5 is proven for valuation ideals associated to an Abhyankar valuation in a regular local ring which is essentially of finite type over a field by Ein, Lazarsfeld and Smith in [15], for general families of m_R -primary ideals when R is a regular local ring containing a field by Mustașă in [31] and when R is a local domain which is essentially of finite type over an algebraically closed field k with $R/m_R = k$ by Lazarsfeld and Ein in Theorem 3.8 [28]. All of these assumptions are necessary in the proof in [28]. The volume = multiplicity formula is proven when R is regular or R is analytically unramified with perfect residue field in Theorem 6.5 [6].

We give, in Theorems 6.1 - Theorem 6.5, some formulas showing that limits of the epsilon multiplicity type exist in analytically unramified local rings. We extend results of [6], where it is assumed that R is equicharacteristic with perfect residue field. Epsilon multiplicity is defined as a limsup by Ulrich and Validashti in [39] and by Kleiman, Ulrich and Validashti in [25]. We also prove an asymptotic formula on multiplicities proposed by Herzog, Puthenpurakal and Verma [20] on analytically unramified local rings. A weaker version of this result is proven in [6]. A general proof of the existence of epsilon multiplicities for torsion free finite rank modules over an analytically unramified local ring is given in Theorem 3.6 of [8] by developing the methods of this paper. In the case of modules over local rings essentially of finite type over an algebraically closed field, this is proven for modules which are locally free over the punctured spectrum in [24] and for more general modules in [5], using different methods.

1.2. Kodaira-Iitaka dimension and growth rate of graded linear series. Before discussing limits of graded linear series we need to define the Kodaira-Iitaka dimension of a graded linear series. This concept is defined classically by several equivalent conditions for normal projective varieties. However, these conditions are no longer equivalent for more general proper schemes, so we must make an appropriate choice in our definition.

Suppose that X is a d -dimensional proper scheme over a field k . A graded linear series $L = \bigoplus_{n \geq 0} L_n$ on X is a graded k -subalgebra of a section ring $\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n)$ of a line bundle \mathcal{L} on X . We define the Kodaira-Iitaka dimension $\kappa(L)$ from the maximal number of algebraically independent forms in L (the complete definition is given in Section 7). This definition agrees with the classical one for normal projective varieties. When X is a projective scheme, the Kodaira-Iitaka dimension of L is $-\infty$ if the Krull dimension of L is 0, and is one less than the Krull dimension of L if the Krull dimension is positive (Lemma 7.2).

Suppose that \mathcal{L} is a line bundle on a normal projective variety X . The index $m(\mathcal{L})$ of \mathcal{L} is defined to be the least common multiple of the positive integers n such that $\Gamma(X, \mathcal{L}^n) \neq 0$. The theorem of Iitaka, Theorem 10.2 [21], tells us that if $\kappa(\mathcal{L}) = -\infty$ then $\dim_k \Gamma(X, \mathcal{L}^n) = 0$ for all positive integers n and if $\kappa(\mathcal{L}) \geq 0$ then there exist positive constants $0 < a < b$ such that

$$(2) \quad an^{\kappa(L)} < \dim_k \Gamma(X, \mathcal{L}^{m(\mathcal{L})n}) < bn^{\kappa(L)}$$

for $n \gg 0$. Thus, with the assumption that X is a normal projective variety, the Kodaira-Iitaka dimension is the growth rate of $\dim_k \Gamma(X, \mathcal{L}^n)$. Equation (2) continues to hold for graded linear series on a proper variety X over a field (this is stated in (45)). However, when X is not integral, (2) may not hold. In fact, the rate of growth has little meaning on nonreduced schemes. In Section 12, it is shown that a graded linear series L on a nonreduced d -dimensional projective scheme with $\kappa(L) = -\infty$ can grow like n^d (Example 12.1) or can oscillate wildly between 0 and n^d (Theorem 12.2 and Example 12.3) so that there is no growth rate.

In fact, it is quite easy to construct badly behaved examples with $\kappa(L) = -\infty$, since in this case the condition that $L_m L_n \subset L_{m+n}$ required for a graded linear series may be vacuous.

1.3. Limits of graded families of linear series.

Let k be a field. From (2) and (45) we have that both $\liminf_{n \rightarrow \infty} \frac{\dim_k L_{mn}}{n^{\kappa(L)}}$ and $\limsup_{n \rightarrow \infty} \frac{\dim_k L_{mn}}{n^{\kappa(L)}}$ exist for a graded linear series L on a proper variety. The remarkable fact is that they actually exist as a common limit on a proper variety.

Suppose that L is a graded linear series on a proper variety X over a field k . The *index* $m = m(L)$ of L is defined as the index of groups

$$m = [\mathbb{Z} : G]$$

where G is the subgroup of \mathbb{Z} generated by $\{n \mid L_n \neq 0\}$.

Theorem 1.2. (Theorem 8.1) *Suppose that X is a d -dimensional proper variety over a field k , and L is a graded linear series on X with Kodaira-Iitaka dimension $\kappa = \kappa(L) \geq 0$. Let $m = m(L)$ be the index of L . Then*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{nm}}{n^{\kappa}}$$

exists.

In particular, from the definition of the index, we have that the limit

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^{\kappa}}$$

exists, whenever n is constrained to lie in an arithmetic sequence $a + bm$ ($m = m(L)$ and a an arbitrary but fixed constant), as $\dim_k L_n = 0$ if $m \nmid n$.

An example of a big line bundle where the limit in Theorem 1.2 is an irrational number is given in Example 4 of Section 7 of the author's paper [12] with Srinivas.

Theorem 1.2 is proven for big line bundles on a nonsingular variety over an algebraically closed field of characteristic zero by Lazarsfeld (Example 11.4.7 [27]) using Fujita approximation (Fujita, [16]). This result is extended by Takagi using De Jong's theory of alterations [13] to hold on nonsingular varieties over algebraically fields of all characteristics $p \geq 0$. Theorem 1.2 has been proven by Okounkov [33] for section rings of ample line bundles, Lazarsfeld and Mustaă [28] for section rings of big line bundles, and for graded linear series by Kaveh and Khovanskii [22] when k is an algebraically closed field. A local form of this result is given by Fulger in [17]. These last proofs use an ingenious method introduced by Okounkov to reduce to a problem of counting points in an integral semi-group. All of these proofs require the assumption that k is algebraically closed. In this paper we establish Theorem 1.2 over an arbitrary ground field k (in Theorem 8.1). We deduce Fujita approximation over an arbitrary field in Theorem 8.5.

It is worth remarking that when k is an arbitrary field and X is geometrically integral over k we easily obtain that the limit of Theorem 1.2 exists by making the base change to $\overline{X} = X \times_k \overline{k}$ where \overline{k} is an algebraic closure of k . Then $\dim_k L_n = \dim_{\overline{k}} \overline{L}_n$ where $\overline{L} = \bigoplus_{n \geq 0} \overline{L}_n$ is the graded linear series on \overline{X} with $\overline{L}_n = L_n \otimes_k \overline{k}$. The scheme \overline{X} is a complete \overline{k} variety (it is integral) since X is geometrically integral. Thus the conclusions of Theorem 1.2 are valid for \overline{L} (on the complete variety \overline{X} over the algebraically closed field \overline{k}) so that the limit for L (on X over k) exists as well. This observation is exploited by Boucksom and Chen in [1] where some limits on geometrically integral arithmetic varieties are computed. However, this argument is not applicable when X is not geometrically integral. The most dramatic difficulty can occur when k is not perfect, as there exist simple examples of irreducible projective varieties which are not even generically reduced after taking the base change to the algebraic closure (we give a simple example below). In Theorem 10.3 it is shown that for general graded linear series the limit does not always exist if X is not generically reduced.

We now give an example, showing that even if X is normal and k is algebraically closed in the function field of X , then $X \times_k \overline{k}$ may not be generically reduced, where \overline{k} is an algebraic closure of k . Let p be a prime number, F_p be the field with p elements and let $k = F_p(s, t, u)$ be a rational function field in three variables over F_p . Let R be the local ring $R = (k[x, y, z]/(sx^p + ty^p + uz^p))_{(x, y, z)}$ with maximal ideal m_R . R is the localization of $T = F_p[s, t, u, x, y, z]/(sx^p + ty^p + uz^p)$ at the ideal (x, y, z) , since $F_p[s, t, u] \cap (x, y, z) = (0)$. T is nonsingular in codimension 1 by the Jacobian criterion over the perfect field F_p , and so T is normal by Serre's criterion. Thus R is normal since it is a localization of T . Let k' be the algebraic closure of k in the quotient field K of R . Then $k' \subset R$ since R is normal. $R/m_R \cong k$ necessarily contains k' , so $k = k'$. However, we have that $R \otimes_k \overline{k}$ is generically not reduced, if \overline{k} is an algebraically closure of k . Now taking X to be a normal projective model of K over k such that R is the local ring of a closed point of X , we have the desired example. In fact, we have that k is algebraically closed in K , but $K \otimes_k \overline{k}$ has nonzero nilpotent elements.

The statement of Theorem 1.2 generalizes very nicely to reduced proper k -schemes, as we establish in Theorem 9.2.

Theorem 1.3. (Theorem 9.2) *Suppose that X is a reduced proper scheme over a field k . Let L be a graded linear series on X with Kodaira-Iitaka dimension $\kappa = \kappa(L) \geq 0$. Then*

there exists a positive integer r such that

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{a+nr}}{n^\varkappa}$$

exists for any fixed $a \in \mathbb{N}$.

The theorem says that

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^\varkappa}$$

exists if n is constrained to lie in an arithmetic sequence $a + br$ with r as above, and for some fixed a . The conclusions of the theorem are a little weaker than the conclusions of Theorem 1.2 for varieties. In particular, the index $m(L)$ has little relevance on reduced but nonirreducible schemes (as shown by the example after Theorem 10.2 and Example 9.5).

Now we turn to the case of nonreduced proper schemes. We begin by returning to our discussion of the relationship between growth rates and the Kodaira-Iitaka dimension of graded linear series, which is much more subtle on nonreduced schemes.

Suppose that X is a proper scheme over a field k . Let \mathcal{N}_X be the nilradical of X . Suppose that L is a graded linear series on X . Then by Theorem 10.2, there exists a positive constant γ such that $\dim_k L_n < \gamma n^e$ where

$$(3) \quad e = \max\{\varkappa(L), \dim \mathcal{N}_X\}.$$

This is the best bound possible. It is shown in Theorem 10.5 that if X is a nonreduced projective k -scheme, then for any $s \in \mathbb{N} \cup \{-\infty\}$ with $s \leq \dim \mathcal{N}_X$, there exists a graded linear series L on X with $\varkappa(L) = s$ and a constant $\alpha > 0$ such that

$$\alpha n^{\dim \mathcal{N}_X} < \dim_k L_n$$

for all $n \gg 0$.

It follows from Theorem 10.3 that if X is a proper k -scheme with $r = \dim \mathcal{N}_X \geq s$, then there exists a graded linear series L of Kodaira-Iitaka dimension $\varkappa(L) = s$ such that the limit

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^r}$$

does not exist, even when n is constrained to lie in *any* arithmetic sequence. However, we prove in Theorem 10.2 that if L is a graded linear series on a proper scheme X over a field k , with $\varkappa(L) > \dim \mathcal{N}_X$, then there exists a positive integer r such that

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{a+nr}}{n^{\varkappa(L)}}$$

exists for any fixed $a \in \mathbb{N}$.

This is the strongest statement on limits that is true. In fact, the existence of all such limits characterizes the dimension of the nilradical, at least on projective schemes. We show this in the following theorem.

Theorem 1.4. (Theorem 10.6) *Suppose that X is a d -dimensional projective scheme over a field k with $d > 0$. Let \mathcal{N}_X be the nilradical of X . Let $\alpha \in \mathbb{N}$. Then the following are equivalent:*

- 1) *For every graded linear series L on X with $\alpha \leq \varkappa(L)$, there exists a positive integer r such that*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{a+nr}}{n^{\varkappa(L)}}$$

exists for every positive integer a .

- 2) For every graded linear series L on X with $\alpha \leq \varkappa(L)$, there exists an arithmetic sequence $a + nr$ (for fixed r and a depending on L) such that

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{a+nr}}{n^{\varkappa(L)}}$$

exists.

- 3) The nilradical \mathcal{N}_X of X satisfies $\dim \mathcal{N}_X < \alpha$.

If X is a proper k -scheme of dimension $d = 0$ which is not irreducible, then the conclusions of Theorem 10.6 are true for X . This follows from Section 11. However, 2) implies 3) does not hold if X is irreducible of dimension 0. In fact (Proposition 11.1) if X is irreducible of dimension 0, and L is a graded linear series on X with $\varkappa(L) = 0$, then there exists a positive integer r such that the limit $\lim_{n \rightarrow \infty} \dim_k L_{a+nr}$ exists for every positive integer a .

1.4. Volumes of line bundles. The volume of a line bundle \mathcal{L} on a d -dimensional proper variety X is the limsup

$$(4) \quad \text{Vol}(\mathcal{L}) = \limsup_{n \rightarrow \infty} \frac{h^0(X, \mathcal{L}^n)}{n^d/d!}.$$

There has been much progress of our understanding of the volume as a function on the big cone in $N^1(X)$ on a projective variety X over an algebraically closed field (where (4) is actually a limit). Much of the theory is explained in [27], where extensive references are given. Volume is continuous on $N^1(X)$ but is not twice differentiable on all of $N^1(X)$ (as shown in an example of Ein Lazarsfeld, Mustaă, Nakamaye and Popa, [14]). Boucksom, Favre and Jonsson [2] have shown that the volume is \mathcal{C}^1 -differentiable on the big cone of $N^1(X)$ (when X is a proper variety over an algebraically closed field of characteristic zero). This theorem is proven for a proper variety over an arbitrary field in [7]. The Fujita approximation type theorem Theorem 8.5, which is valid over an arbitrary field, is an ingredient of the proof. Interpretation of the directional derivative in terms of intersection products and many applications are given in [2], [14], [28] and [7].

The starting point of the theory of volume on nonreduced schemes is to determine if the limsup defined in (4) exists as a limit. In Theorem 10.7, it is shown that the volume of a line bundle always exists on a d -dimensional proper scheme X over a field k with $\dim \mathcal{N}_X < d$ (as explained earlier, this result was known on varieties over an algebraically closed field). We see from Theorem 10.3 and Example 12.1 that the limit does not always exist for graded linear series L .

2. NOTATION AND CONVENTIONS

m_R will denote the maximal ideal of a local ring R . $Q(R)$ will denote the quotient field of a domain R . $\ell_R(N)$ will denote the length of an R -module N . \mathbb{Z}_+ denotes the positive integers and \mathbb{N} the nonnegative integers. Suppose that $x \in \mathbb{R}$. $\lceil x \rceil$ is the smallest integer n such that $x \leq n$. $\lfloor x \rfloor$ is the largest integer n such that $n \leq x$.

We recall some notation on multiplicity from Chapter VIII, Section 10 of [40], Section V-2 [35] and Section 4.6 [4]. Suppose that (R, m_R) is a (Noetherian) local ring, N is a finitely generated R -module with $r = \dim N$ and a is an ideal of definition of R . Then

$$e_a(N) = \lim_{k \rightarrow \infty} \frac{\ell_R(N/a^k N)}{k^r/r!}.$$

We write $e(a) = e_a(R)$.

If $s \geq r = \dim N$, then we define

$$e_s(a, N) = \begin{cases} e_a(N) & \text{if } \dim N = s \\ 0 & \text{if } \dim N < s. \end{cases}$$

A local ring is analytically unramified if its completion is reduced. In particular, a reduced excellent local ring is analytically unramified.

We will denote the maximal ideal of a local ring R by m_R . If ν is a valuation of a field K , then we will write V_ν for the valuation ring of ν , and m_ν for the maximal ideal of V_ν . We will write Γ_ν for the value group of ν . If A and B are local rings, we will say that B dominates A if $A \subset B$ and $m_B \cap A = m_A$.

The dimension of an R -module M is $\dim M = \dim R/\text{ann}(M)$.

We use the notation of Hartshorne [19]. For instance, a variety is required to be integral. If \mathcal{F} is a coherent sheaf on a Noetherian scheme, then $\dim \mathcal{F}$ will denote the dimension of the support of \mathcal{F} , with $\dim \mathcal{F} = -\infty$ if $\mathcal{F} = 0$.

Suppose that X is a scheme. The nilradical of X is the ideal sheaf \mathcal{N}_X on X which is the kernel of the natural surjection $\mathcal{O}_X \rightarrow \mathcal{O}_{X_{\text{red}}}$ where X_{red} is the reduced scheme associated to X . $(\mathcal{N}_X)_\eta$ is the nilradical of the local ring $\mathcal{O}_{X,\eta}$ for all $\eta \in X$.

3. CONES ASSOCIATED TO SEMIGROUPS

In this section, we summarize some results of Okounkov [33], Lazarsfeld and Mustařă [28] and Kaveh and Khovanskii [22].

Suppose that S is a subsemigroup of $\mathbb{Z}^d \times \mathbb{N}$ which is not contained in $\mathbb{Z}^d \times \{0\}$. Let $L(S)$ be the subspace of \mathbb{R}^{d+1} which is generated by S , and let $M(S) = L(S) \cap (\mathbb{R}^d \times \mathbb{R}_{\geq 0})$.

Let $\text{Con}(S) \subset L(S)$ be the closed convex cone which is the closure of the set of all linear combinations $\sum \lambda_i s_i$ with $s_i \in S$ and $\lambda_i \geq 0$.

S is called *strongly nonnegative* (Section 1.4 [22]) if $\text{Cone}(S)$ intersects $\partial M(S)$ only at the origin (this is equivalent to being strongly admissible (Definition 1.9 [22]) since with our assumptions, $\text{Cone}(S)$ is contained in $\mathbb{R}^d \times \mathbb{R}_{\geq 0}$, so the ridge of S must be contained in $\partial M(S)$). In particular, a subsemigroup of a strongly negative semigroup is itself strongly negative.

We now introduce some notation from [22]. Let

$$S_k = S \cap (\mathbb{R}^d \times \{k\}).$$

$$\Delta(S) = \text{Con}(S) \cap (\mathbb{R}^d \times \{1\}) \text{ (the Newton-Okounkov body of } S).$$

$$q(S) = \dim \partial M(S).$$

$$G(S) \text{ be the subgroup of } \mathbb{Z}^{d+1} \text{ generated by } S.$$

$$m(S) = [\mathbb{Z} : \pi(G(S))] \text{ where } \pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R} \text{ be projection onto the last factor.}$$

$$\text{ind}(S) = [\partial M(S)_\mathbb{Z} : G(S) \cap \partial M(S)_\mathbb{Z}] \text{ where}$$

$$\partial M(S)_\mathbb{Z} := \partial M(S) \cap \mathbb{Z}^{d+1} = M(S) \cap (\mathbb{Z}^d \times \{0\}).$$

$\text{vol}_{q(S)}(\Delta(S))$ is the integral volume of $\Delta(S)$. This volume is computed using the translation of the integral measure on $\partial M(S)$.

S is strongly negative if and only if $\Delta(S)$ is a compact set. If S is strongly negative, then the dimension of $\Delta(S)$ is $q(S)$.

Theorem 3.1. (*Kaveh and Khovanskii*) Suppose that S is strongly nonnegative. Then

$$\lim_{k \rightarrow \infty} \frac{\#S_{m(S)k}}{k^{q(S)}} = \frac{\text{vol}_{q(S)}(\Delta(S))}{\text{ind}(S)}.$$

This is proven in Corollary 1.16 [22].

With our assumptions, we have that $S_n = \emptyset$ if $m(S) \nmid n$ and the limit is positive, since $\text{vol}_q(\Delta(S)) > 0$.

Theorem 3.2. (*Okounkov, Section 3 [33], Lazarsfeld and Mustař, Proposition 2.1 [28]*) Suppose that a subsemigroup S of $\mathbb{Z}^d \times \mathbb{N}$ satisfies the following two conditions:

- (5) *There exist finitely many vectors $(v_i, 1)$ spanning a semigroup $B \subset \mathbb{N}^{d+1}$ such that $S \subset B$*

and

(6)
$$G(S) = \mathbb{Z}^{d+1}.$$

Then

$$\lim_{n \rightarrow \infty} \frac{\#S_n}{n^d} = \text{vol}(\Delta(S)).$$

Proof. S is strongly nonnegative since B is strongly nonnegative, so Theorem 3.1 holds.

$G(S) = \mathbb{Z}^{d+1}$ implies $L(S) = \mathbb{R}^{d+1}$, so $M(S) = \mathbb{R}^d \times \mathbb{R}_{\geq 0}$, $\partial M(S) = \mathbb{R}^d \times \{0\}$ and $q(S) = \dim \partial M(S) = d$. We thus have $m(S) = 1$ and $\text{ind}(S) = 1$. \square

Theorem 3.3. Suppose that S is strongly nonnegative. Fix $\varepsilon > 0$. Then there is an integer $p = p_0(\varepsilon)$ such that if $p \geq p_0$, then the limit

$$\lim_{n \rightarrow \infty} \frac{\#(n * S_{pm(S)})}{n^{q(S)} p^{q(S)}} \geq \frac{\text{vol}_{q(S)} \Delta(S)}{\text{ind}(S)} - \varepsilon$$

exists, where

$$n * S_{pm(S)} = \{x_1 + \cdots + x_n \mid x_1, \dots, x_n \in S_{pm(S)}\}.$$

Proof. Let $m = m(S)$ and $q = q(S)$. Let $S^{[pm]} = \cup_{n=1}^{\infty} (n * S_{pm(S)})$ be the subsemigroup of S generated by S_{pm} . For $p \gg 0$, we have that $L(S^{[pm]}) = L(S)$ so $m(S^{[pm]}) = pm$ and $q(S^{[pm]}) = q$.

Suppose that v_1, \dots, v_r generate $G(S) \cap \partial M(S)_{\mathbb{Z}}$. For $1 \leq i \leq r$, there exist a_i, b_i, n_i such that $v_i = (a_i, n_i m) - (b_i, n_i m)$ with $(a_i, n_i m), (b_i, n_i m) \in S_{n_i m}$. There exist $b > 0$ and c, c' such that $(c, mb) \in S$ and $(c', m(b+1)) \in S$. bm divides $n_i m + n_i(b-1)(b+1)m$ and

$$v_i = [(a_i, n_i m) + n_i(b-1)(c', (b+1)m)] - [(b_i, n_i m) + n_i(b-1)(c', (b+1)m)],$$

so we may assume that b divides n_i for all i . Thus $v_1, \dots, v_r \in G(S^{[nm]})$ where $n = \max\{n_i\}$, and $v_1, \dots, v_r \in G(S^{[pm]})$ whenever $p \geq (b-1)b + n$. Thus

(7)
$$\text{ind}(S^{[pm]}) = \text{ind}(S)$$

whenever $p \gg 0$. We have that

(8)
$$\lim_{p \rightarrow \infty} \frac{\text{vol}_q(\Delta(S^{[pm]}))}{p^q} = \text{vol}_q(\Delta(S)).$$

By Theorem 3.1,

(9)
$$\lim_{n \rightarrow \infty} \frac{\#(n * S_{pm})}{n^q} = \frac{\text{vol}_q(\Delta(S^{[pm]}))}{\text{ind}(S^{[pm]})}.$$

The theorem now follows from (7), (8), (9). \square

We obtain the following result.

Theorem 3.4. (Proposition 3.1 [28]) Suppose that a subsemigroup S of $\mathbb{Z}^d \times \mathbb{N}$ satisfies (5) and (6). Fix $\varepsilon > 0$. Then there is an integer $p_0 = p_0(\varepsilon)$ such that if $p \geq p_0$, then the limit

$$\lim_{k \rightarrow \infty} \frac{\#(k * S_p)}{k^d p^d} \geq \text{vol}(\Delta(S)) - \varepsilon$$

exists.

4. ASYMPTOTIC THEOREMS ON LENGTHS

Definition 4.1. A graded family of ideals $\{I_i\}$ in a ring R is a family of ideals indexed by the natural numbers such that $I_0 = R$ and $I_i I_j \subset I_{i+j}$ for all i, j . If R is a local ring and I_i is m_R -primary for $i > 0$, then we will say that $\{I_i\}$ is a graded family of m_R -primary ideals.

The following theorem is proven with the further assumptions that R is equicharacteristic with perfect residue field in [6].

Theorem 4.2. Suppose that R is an analytically irreducible local domain of dimension d and $\{I_i\}$ is a graded family of m_R -primary ideals in R . Then

$$\lim_{i \rightarrow \infty} \frac{\ell_R(R/I_i)}{i^d}$$

exists.

Corollary 4.3. Suppose that R is an analytically irreducible local domain of dimension $d > 0$ and $\{I_i\}$ is a graded family of ideals in R such that there exists a positive number c such that $m_R^c \subset I_1$. Then

$$\lim_{i \rightarrow \infty} \frac{\ell_R(R/I_i)}{i^d}$$

exists.

Proof. The assumption $m_R^c \subset I_1$ implies that either I_n is m_R -primary for all positive n , or there exists $n_0 > 1$ such that $I_{n_0} = R$. In the first case, the corollary follows from Theorem 4.2. In the second case, $m_R^{cn} \subset I_n$ for all $n \geq n_0$, so $\ell_R(R/I_i)$ is actually bounded. \square

We now give the proof of Theorem 4.2.

Since I_1 is m_R -primary, there exists $c \in \mathbb{Z}_+$ such that

$$(10) \quad m_R^c \subset I_1.$$

Let \hat{R} be the m_R -adic completion of R . Since the I_n are m_R -primary, we have that $R/I_n \cong \hat{R}/I_n \hat{R}$ and $\ell_R(R/I_n) = \ell_{\hat{R}}(\hat{R}/I_n \hat{R})$ for all n . We may thus assume that R is an excellent domain. Let $\pi : X \rightarrow \text{spec}(R)$ be the normalization of the blow up of m_R . X is of finite type over R since R is excellent. Since $\pi^{-1}(m_R)$ has codimension 1 in X and X is normal, there exists a closed point $x \in X$ such that the local ring $\mathcal{O}_{X,x}$ is a regular local ring. Let S be this local ring. S is a regular local ring which is essentially of finite type and birational over R (R and S have the same quotient field).

Let y_1, \dots, y_d be a regular system of parameters in S . Let $\lambda_1, \dots, \lambda_d$ be rationally independent real numbers, such that

$$(11) \quad \lambda_i \geq 1 \text{ for all } i.$$

We define a valuation ν on $Q(R)$ which dominates S by prescribing

$$\nu(y_1^{a_1} \cdots y_d^{a_d}) = a_1 \lambda_1 + \cdots + a_d \lambda_d$$

for $a_1, \dots, a_d \in \mathbb{Z}_+$, and $\nu(\gamma) = 0$ if $\gamma \in S$ has nonzero residue in S/m_S .

Let C be a coefficient set of S . Since S is a regular local ring, for $r \in \mathbb{Z}_+$ and $f \in S$, there is a unique expression

$$(12) \quad f = \sum s_{i_1, \dots, i_d} y_1^{i_1} \cdots y_d^{i_d} + g_r$$

with $g_r \in m_S^r$, $s_{i_1, \dots, i_d} \in S$ and $i_1 + \dots + i_d < r$ for all i_1, \dots, i_d appearing in the sum. Take r so large that $r > i_1 \lambda_1 + \dots + i_d \lambda_d$ for some term with $s_{i_1, \dots, i_d} \neq 0$. Then define

$$(13) \quad \nu(f) = \min\{i_1 \lambda_1 + \dots + i_d \lambda_d \mid s_{i_1, \dots, i_d} \neq 0\}.$$

This definition is well defined, and we calculate that $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$ and $\nu(fg) = \nu(f) + \nu(g)$ (by the uniqueness of the minimal value term in the expansion (12)) for all $0 \neq f, g \in S$. Thus ν is a valuation. Let V_ν be the valuation ring of ν (in $Q(R)$). The value group Γ_ν of V_ν is the (nondiscrete) ordered subgroup $\mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_d$ of \mathbb{R} . Since there is a unique monomial giving the minimum in (13), we have that the residue field of V_ν is S/m_S .

Let $k = R/m_R$ and $k' = S/m_S = V_\nu/m_\nu$. Since S is essentially of finite type over R , we have that $[k' : k] < \infty$.

For $\lambda \in \mathbb{R}$, define ideals K_λ and K_λ^+ in V_ν by

$$K_\lambda = \{f \in Q(R) \mid \nu(f) \geq \lambda\}$$

and

$$K_\lambda^+ = \{f \in Q(R) \mid \nu(f) > \lambda\}.$$

We follow the usual convention that $\nu(0) = \infty$ is larger than any element of \mathbb{R} . By Lemma 4.3 [6], we have the following formula. The assumption that R is analytically irreducible is necessary for the validity of the formula.

$$(14) \quad \text{There exists } \alpha \in \mathbb{Z}_+ \text{ such that } K_{\alpha n} \cap R \subset m_R^n \text{ for all } n \in \mathbb{N}.$$

Suppose that $I \subset R$ is an ideal and $\lambda \in \Gamma_\nu$ is nonnegative. Then we have inclusions of k -vector spaces

$$I \cap K_\lambda / I \cap K_\lambda^+ \subset K_\lambda / K_\lambda^+.$$

Since K_λ / K_λ^+ is isomorphic to k' , we conclude that

$$(15) \quad \dim_k I \cap K_\lambda / I \cap K_\lambda^+ \leq [k' : k].$$

Let $\beta = \alpha c \in \mathbb{Z}_+$, where c is the constant of (10), and α is the constant of (14), so that for all $i \in \mathbb{Z}_+$,

$$(16) \quad K_{\beta i} \cap R = K_{\alpha c i} \cap R \subset m_R^{ic} \subset I_i.$$

For $t \geq 1$, define

$$\Gamma^{(t)} = \left\{ \begin{array}{l} (n_1, \dots, n_d, i) \in \mathbb{N}^{d+1} \mid \dim_k I_i \cap K_{n_1 \lambda_1 + \dots + n_d \lambda_d} / I_i \cap K_{n_1 \lambda_1 + \dots + n_d \lambda_d}^+ \geq t \\ \text{and } n_1 + \dots + n_d \leq \beta i \end{array} \right\},$$

and

$$\hat{\Gamma}^{(t)} = \left\{ \begin{array}{l} (n_1, \dots, n_d, i) \in \mathbb{N}^{d+1} \mid \dim_k R \cap K_{n_1 \lambda_1 + \dots + n_d \lambda_d} / R \cap K_{n_1 \lambda_1 + \dots + n_d \lambda_d}^+ \geq t \\ \text{and } n_1 + \dots + n_d \leq \beta i \end{array} \right\}.$$

Let $\lambda = n_1 \lambda_1 + \dots + n_d \lambda_d$ be such that $n_1 + \dots + n_d \leq \beta i$. Then

$$(17) \quad \dim_k K_\lambda \cap I_i / K_\lambda^+ \cap I_i = \#\{t \mid (n_1, \dots, n_d, i) \in \Gamma^{(t)}\}.$$

Lemma 4.4. Suppose that $t \geq 1$, $0 \neq f \in I_i$, $0 \neq g \in I_j$ and

$$\dim_k I_i \cap K_{\nu(f)}/I_i \cap K_{\nu(f)}^+ \geq t.$$

Then

$$(18) \quad \dim_k I_{i+j} \cap K_{\nu(fg)}/I_{i+j} \cap K_{\nu(fg)}^+ \geq t.$$

In particular, when nonempty, $\Gamma^{(t)}$ and $\hat{\Gamma}^{(t)}$ are subsemigroups of the semigroup \mathbb{Z}^{d+1} .

Proof. There exist $f_1, \dots, f_t \in I_i \cap K_{\nu(f)}$ such that their classes are linearly independent over k in $I_i \cap K_{\nu(f)}/I_i \cap K_{\nu(f)}^+$. We will show that the classes of gf_1, \dots, gf_t in

$$I_{i+j} \cap K_{\nu(fg)}/I_{i+j} \cap K_{\nu(fg)}^+$$

are linearly independent over k .

Suppose that $a_1, \dots, a_t \in k$ are such that the class of $a_1gf_1 + \dots + a_tgf_t$ in $I_{i+j} \cap K_{\nu(fg)}/I_{i+j} \cap K_{\nu(fg)}^+$ is zero. Then $\nu(a_1gf_1 + \dots + a_tgf_t) > \nu(fg)$, whence $\nu(a_1f_1 + \dots + a_tf_t) > \nu(f)$, so $a_1f_1 + \dots + a_tf_t \in I_i \cap K_{\nu(f)}^+$. Thus $a_1 = \dots = a_t = 0$, since the classes of f_1, \dots, f_t are linearly independent over k in $I_i \cap K_{\nu(f)}/I_i \cap K_{\nu(f)}^+$. \square

From (16), and since $n_1\lambda_1 + \dots + n_d\lambda_d < \beta i$ implies $n_1 + \dots + n_d < \beta i$ by (11), we have that

$$(19) \quad \begin{aligned} \ell_R(R/I_i) &= \ell_R(R/K_{\beta i} \cap R) - \ell_R(I_i/K_{\beta i} \cap I_i) \\ &= \dim_k \left(\bigoplus_{0 \leq \lambda < \beta i} K_\lambda \cap R/K_\lambda^+ \cap R \right) - \dim_k \left(\bigoplus_{0 \leq \lambda < \beta i} K_\lambda \cap I_i/K_\lambda^+ \cap I_i \right) \\ &= \left(\sum_{t=1}^{[k':k]} \# \hat{\Gamma}_i^{(t)} \right) - \left(\sum_{t=1}^{[k':k]} \# \Gamma_i^{(t)} \right), \end{aligned}$$

where $\Gamma_i^{(t)} = \Gamma^{(t)} \cap (\mathbb{N}^d \times \{i\})$ and $\hat{\Gamma}_i^{(t)} = \hat{\Gamma}^{(t)} \cap (\mathbb{N}^d \times \{i\})$. Since R is Noetherian, there are only finitely many values of ν on R which are $\leq \beta i$,

For $0 \neq f \in R$, define

$$\varphi(f) = (n_1, \dots, n_d) \in \mathbb{N}^d$$

if $\nu(f) = n_1\lambda_1 + \dots + n_d\lambda_d$. We have that $\varphi(fg) = \varphi(f) + \varphi(g)$.

Lemma 4.5. Suppose that $t \geq 1$ and $\Gamma^{(t)} \not\subseteq (0)$. Then $\Gamma^{(t)}$ satisfies equations (5) and (6).

Proof. Let $\{e_i\}$ be the standard basis of \mathbb{Z}^{d+1} . The semigroup

$$B = \{(n_1, \dots, n_d, i) \mid (n_1, \dots, n_d) \in \mathbb{N}^d \text{ and } n_1 + \dots + n_d \leq \beta i\}$$

is generated by $B \cap (\mathbb{N}^d \times \{1\})$ and contains $\Gamma^{(t)}$, so (5) holds.

By assumption, there exists $r \geq 1$ and $0 \neq h \in I_r$ such that $(\varphi(h), r) \in \Gamma^{(t)}$.

There exists $0 \neq u \in I_1$. Write $y_i = \frac{f_i}{g_i}$ with $f_i, g_i \in R$ for $1 \leq i \leq d$. Then $hf_i, hg_i \in I_r$. There exists $c' \in \mathbb{Z}_+$ such that $c' \geq c$ and $u, hf_i, hg_i \notin m_R^{c'}$ for $1 \leq i \leq d$. We may replace c with c' in (10). Then $(\varphi(hf_i), r), (\varphi(hg_i), r) \in \Gamma_r^{(t)} = \Gamma^{(t)} \cap (\mathbb{N}^d \times \{r\})$ for $1 \leq i \leq d$, by (18) (with $f_i, g_i \in I_0 = R$) since hf_i and hg_i all have values $n_1\lambda_1 + \dots + n_d\lambda_d < \beta r$, so that $n_1 + \dots + n_d < \beta r$. We have that $\varphi(y_i) = \varphi(hf_i) - \varphi(hg_i) = \varphi(y_i) = e_i$ for $1 \leq i \leq d$. Thus

$$(e_i, 0) = (\varphi(hf_i), r) - (\varphi(hg_i), r) \in G(\Gamma^{(t)})$$

for $1 \leq i \leq d$. $(\varphi(uh), r+1) \in \Gamma^{(t)}$ by (18) and since $\nu(u) \leq \beta$, so that $(\varphi(u), 1) \in G(\Gamma^{(t)})$, so $e_{d+1} \in G(\Gamma^{(t)})$. Thus $G(\Gamma^{(t)}) = \mathbb{Z}^{d+1}$ and (6) holds. \square

The same argument proves the following lemma.

Lemma 4.6. *Suppose that $t \geq 1$ and $\hat{\Gamma}^{(t)} \not\subset (0)$. Then $\hat{\Gamma}^{(t)}$ satisfies equations (5) and (6).*

By Theorem 3.2,

$$(20) \quad \lim_{i \rightarrow \infty} \frac{\#\Gamma_i^{(t)}}{i^d} = \text{vol}(\Delta(\Gamma^{(t)}))$$

and

$$(21) \quad \lim_{i \rightarrow \infty} \frac{\#\hat{\Gamma}_i^{(t)}}{i^d} = \text{vol}(\Delta(\hat{\Gamma}^{(t)})).$$

We obtain the conclusions of Theorem 4.2 from equations (19), (20) and (21).

Theorem 4.7. *Suppose that R is a local ring of dimension d , and $\{I_i\}$ is a graded family of m_R -primary ideals in R . Let $N(\hat{R})$ be the nilradical of the m_R -adic completion \hat{R} of R , and suppose that $\dim N(\hat{R}) < d = \dim R$. Then*

$$\lim_{i \rightarrow \infty} \frac{\ell_R(R/I_i)}{i^d}$$

exists.

Proof. Let $N = N(\hat{R})$ and $A = \hat{R}/N$. We have a short exact sequence of \hat{R} -modules

$$0 \rightarrow N/(N \cap I_i \hat{R}) \rightarrow \hat{R}/I_i \hat{R} \rightarrow A/I_i A \rightarrow 0.$$

There exists a number c such that $m_R^c \subset I_1$. Hence $m_R^{ci} N \subset N \cap I_i \hat{R}$ for all i , so that

$$\ell_{\hat{R}}(N/(N \cap I_i \hat{R})) \leq \ell_{\hat{R}}(N/m_R^{ci} N) \leq \alpha i^{\dim N}$$

for some constant α . Hence

$$\lim_{i \rightarrow \infty} \frac{\ell_R(R/I_i)}{i^d} = \lim_{i \rightarrow \infty} \frac{\ell_{\hat{R}}(\hat{R}/I_i \hat{R})}{i^d} = \lim_{i \rightarrow \infty} \frac{\ell_A(A/I_i A)}{i^d}.$$

Let p_1, \dots, p_s be the minimal primes of A , and $A_j = A/p_j$ for $1 \leq j \leq s$. By Lemma 4.8 below,

$$\lim_{i \rightarrow \infty} \frac{\ell_A(A/I_i A)}{i^d} = \sum_{j=1}^s \lim_{i \rightarrow \infty} \frac{\ell_{A_j}(A_j/I_i A_j)}{i^d}$$

which exists by Theorem 4.2 □

Lemma 4.8. *(Lemma 5.1 [6]) Suppose that R is a d -dimensional reduced local ring and $\{I_n\}$ is a graded family of m_R -primary ideals in R . Let $\{p_1, \dots, p_s\}$ be the minimal primes of R , $R_i = R/p_i$, and let S be the ring $S = \bigoplus_{i=1}^s R_i$. Then there exists $\alpha \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$,*

$$\left| \sum_{i=1}^s \ell_{R_i}(R_i/I_n R_i) - \ell_R(R/I_n) \right| \leq \alpha n^{d-1}.$$

5. A NECESSARY AND SUFFICIENT CONDITION FOR LIMITS TO EXIST IN A LOCAL RING

Let $i_1 = 2$ and $r_1 = \frac{i_1}{2}$. For $j \geq 1$, inductively define i_{j+1} so that i_{j+1} is even and $i_{j+1} > 2^j i_j$. Let $r_{j+1} = \frac{i_{j+1}}{2}$. For $n \in \mathbb{Z}_+$, define

$$(22) \quad \sigma(n) = \begin{cases} 1 & \text{if } n = 1 \\ \frac{i_j}{2} & \text{if } i_j \leq n < i_{j+1} \end{cases}$$

Lemma 5.1. *Suppose that $a \in \mathbb{N}$ and $r \in \mathbb{Z}_+$. Then given $m > 0$ and $\varepsilon > 0$, there exists a positive integer $n = a + br$ with $b \in \mathbb{N}$ such that $n > m$ and*

$$\left| \frac{\sigma(n)}{n} - \frac{1}{2} \right| < \varepsilon$$

Proof. Choose j sufficiently large that $i_j > m$, $i_j + r < i_{j+1}$ and

$$(23) \quad \frac{i_j}{2(i_j + k)} > \frac{1}{2} - \varepsilon$$

for $0 \leq k < r$. There exists $n = i_j + k$ with $0 \leq k < r$ in the arithmetic sequence $a + br$.

$$\frac{\sigma(n)}{n} = \frac{i_j}{2n} = \frac{i_j}{2(i_j + k)}.$$

By (23),

$$\frac{1}{2} \geq \frac{i_j}{2(i_j + k)} > \frac{1}{2} - \varepsilon.$$

□

Lemma 5.2. *Suppose that $a \in \mathbb{N}$ and $r \in \mathbb{Z}_+$. Then given $m > 0$ and $\varepsilon > 0$, there exists a positive integer $n = a + br$ with $b \in \mathbb{N}$ such that $n > m$ and*

$$\left| \frac{\sigma(n)}{n} \right| < \varepsilon.$$

Proof. Choose j sufficiently large that $i_j > m + r$, $2^j i_j > r$ and

$$(24) \quad \frac{i_j}{2(2^j i_j - k)} < \varepsilon$$

for $0 < k \leq r$. Let $n = i_{j+1} - k$ with $0 < k \leq r$ in the arithmetic sequence $a + br$.

$$\frac{\sigma(n)}{n} = \frac{i_j}{2n} = \frac{i_j}{2(i_{j+1} - k)}.$$

By (24),

$$0 < \frac{i_j}{2(i_{j+1} - k)} < \varepsilon.$$

□

It follows from the previous two lemmas that the limit

$$(25) \quad \lim_{n \rightarrow \infty} \frac{\sigma(n)}{n}$$

does not exist, even when n is constrained to lie in an arithmetic sequence. The following example shows that limits might not exist on nonreduced local rings.

Example 5.3. Let k be a field, $d > 0$ and R be the nonreduced d -dimensional local ring $R = k[[x_1, \dots, x_d, y]]/(y^2)$. There exists a graded family of m_R -primary ideals $\{I_n\}$ in R such that the limit

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d}$$

does not exist, even when n is constrained to lie in an arithmetic sequence.

Proof. Let $\bar{x}_1, \dots, \bar{x}_d, \bar{y}$ be the classes of x_1, \dots, x_d, y in R . Let N_i be the set of monomials of degree i in the variables $\bar{x}_1, \dots, \bar{x}_d$. Let $\sigma(n)$ be the function defined in (22). Define m_R -primary ideals I_n in R by $I_n = (N_n, \bar{y}N_{n-\sigma(n)})$ for $n \geq 1$ (and $I_0 = R$).

We first verify that $\{I_n\}$ is a graded family of ideals, by showing that $I_m I_n \subset I_{m+n}$ for all $m, n > 0$. This follows since

$$I_m I_n = (N_{m+n}, \bar{y}N_{(m+n)-\sigma(m)}, \bar{y}N_{(m+n)-\sigma(n)})$$

and $\sigma(j) \leq \sigma(k)$ for $k \geq j$.

R/I_n has a k -basis consisting of

$$\{N_i \mid i < n\} \text{ and } \{\bar{y}N_j \mid j < n - \sigma(n)\}.$$

Thus

$$\ell_R(R/I_n) = \binom{n}{d} + \binom{n - \sigma(n)}{d}.$$

does not exist, even when n is constrained to lie in an arithmetic sequence, by (25). \square

Hailong Dao and Ilya Smirnov have communicated to me that they have proven the following theorem.

Theorem 5.4. (Hailong Dao and Ilya Smirnov) Suppose that R is a local ring of dimension $d > 0$ with nilradical $N(R)$. Suppose that for any graded family $\{I_n\}$ of m_R -primary ideals, the limit

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d}$$

exists. Then $\dim N(R) < d$.

Proof. Let $N = N(R)$. Suppose that $\dim N = d$. Let p be a minimal prime of N such that $\dim R/p = d$. Then $N_p \neq 0$, so $p_p \neq 0$ in R_p . p is an associated prime of N , so there exists $0 \neq x \in R$ such that $\text{ann}(x) = p$. $x \in p$, since otherwise $0 = pxR_p = p_p$ which is impossible. In particular, $x^2 = 0$.

Let $f(n) = n - \sigma(n)$ be the function of (22), and define m_R -primary ideals in R by

$$I_n = m_R^n + x m_R^{f(n)}.$$

$\{I_n\}$ is a graded family of ideals in R since

$$I_m I_n = (m_R^{m+n}, x m_R^{(m+n)-\sigma(m)}, x m_R^{(m+n)-\sigma(n)})$$

and $\sigma(j) \leq \sigma(k)$ for $k \geq j$. Let $\bar{R} = R/xR$. We have short exact sequences

$$(26) \quad 0 \rightarrow xR/xR \cap I_n \rightarrow R/I_n \rightarrow \bar{R}/I_n \bar{R} \rightarrow 0.$$

By Artin-Rees, there exists a number k such that $xR \cap m_R^n = m_R^{n-k}(xR \cap m_R^{n-k})$ for $n > k$. Thus $xR \cap m_R^n \subset x m_R^{f(n)}$ for $n \gg 0$ and $xR \cap I_n = x m_R^{f(n)}$ for $n \gg 0$. We have that

$$xR/xR \cap I_n \cong xR/x m_R^{f(n)} \cong R/(\text{ann}(x) + m_R^{f(n)}) \cong R/p + m_R^{f(n)},$$

so that $\ell_R(xR/xR \cap I_n) = P_{R/p}(f(n))$ for $n \gg 0$, where $P_{R/p}(n)$ is the Hilbert-Samuel polynomial of R/p . Hence

$$(27) \quad \lim_{n \rightarrow \infty} \frac{\ell_R(xR/xR \cap I_n)}{n^d} = \frac{e(m_{R/p})}{d!} \lim_{n \rightarrow \infty} \left(\frac{f(n)}{n} \right)^d$$

does not exist by (25). For $n \gg 0$,

$$\ell_R(\overline{R}/I_n \overline{R}) = \ell_R(\overline{R}/m_{\overline{R}}^n) = P_{\overline{R}}(n)$$

where $P_{\overline{R}}(n)$ is the Hilbert-Samuel polynomial of \overline{R} . Since $\dim \overline{R} \leq d$, we have that

$$(28) \quad \lim_{n \rightarrow \infty} \frac{\ell_R(\overline{R}/I_n \overline{R})}{n^d}$$

exists. Thus

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d}$$

does not exist by (26), (27) and (28). \square

Theorem 5.5. *Suppose that R is a local ring of dimension d , and $N(\hat{R})$ is the nilradical of the m_R -adic completion \hat{R} of R . Then the limit*

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d}$$

exists for any graded family $\{I_n\}$ of m_R -primary ideals, if and only if $\dim N(\hat{R}) < d$.

Proof. Sufficiency follows from Theorem 4.7. Necessity follows from Theorem 5.4 if $d > 0$, since a family of $m_{\hat{R}}$ -primary ideals in \hat{R} naturally lifts to a graded family of m_R -primary ideals in R .

In the case when $d = 0$ and $N(\hat{R}) \neq 0$, R is an Artin local ring. Thus there exists some number $0 < t$ such that $m_R^t \neq 0$ but $m_R^{t+1} = 0$. With the notation before (22), let

$$(29) \quad \tau(n) = \begin{cases} 0 & \text{if } i_j \leq n \leq i_{j+1} \text{ and } j \text{ is even} \\ 1 & \text{if } i_j \leq n \leq i_{j+1} \text{ and } j \text{ is odd.} \end{cases}$$

Define a graded family of m_R -primary ideals $\{I_n\}$ in R by $I_n = m_R^{t+\tau(n)}$. Then $\lim_{n \rightarrow \infty} \ell_R(R/I_n)$ does not exist. \square

Corollary 5.6. *Suppose that R is an excellent local ring of dimension d , and $N(R)$ is the nilradical of R . Then the limit*

$$\lim_{i \rightarrow \infty} \frac{\ell_R(R/I_i)}{n^d}$$

exists for any graded family $\{I_i\}$ of m_R -primary ideals, if and only if $\dim N(R) < d$.

Proof. Let $N(\hat{R})$ be the nilradical of \hat{R} . $(\widehat{R/N(R)}) \cong \hat{R}/N(R)\hat{R}$ is reduced since $R/N(R)$ is (by Scholie IV.7.8.3 [18]). Since $N(R)\hat{R} \subset N(\hat{R})$, we have that $N(\hat{R}) = N(R)\hat{R}$. Thus $\text{gr}_{m_{\hat{R}}}(N(\hat{R})) = \text{gr}_{m_R}(N(R))$, so $\dim N(\hat{R}) = \dim N(R)$. Now the corollary follows from Theorem 5.5. \square

Example 5.7. *For any $d \geq 1$, there exists a local domain R of dimension d with a graded family of m_R -primary ideals $\{I_n\}$ such that the limit*

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d}$$

does not exist.

Proof. The example of (E3.2) in [32] is of a local domain R such that the nilradical of \hat{R} has dimension d . The example then follows from Theorem 5.4 by lifting an appropriate graded family of $m_{\hat{R}}$ -primary ideals to R . \square

In Section 4 of [10], a series of examples of graded families of m_R -primary valuation ideals in a regular local ring R of dimension two are given which have asymptotic growth of the rate n^α , where α can be any rational number $1 \leq \alpha \leq 2$. An example, also in a regular local ring of dimension two, with growth rate $n \log_{10}(n)$ is given. Thus we generally do not have a polynomial rate of growth.

6. APPLICATIONS TO ASYMPTOTIC MULTIPLICITIES

In this section, we apply Theorem 4.2 and its method of proof, to generalize some of the applications in [6] to analytically unramified local rings.

Theorem 6.1. *Suppose that R is an analytically unramified local ring of dimension $d > 0$. Suppose that $\{I_i\}$ and $\{J_i\}$ are graded families of ideals in R . Further suppose that $I_i \subset J_i$ for all i and there exists $c \in \mathbb{Z}_+$ such that*

$$(30) \quad m_R^{ci} \cap I_i = m_R^{ci} \cap J_i$$

for all i . Assume that if P is a minimal prime of R then $I_1 \subset P$ implies $I_i \subset P$ for all $i \geq 1$. Then the limit

$$\lim_{i \rightarrow \infty} \frac{\ell_R(J_i/I_i)}{i^d}$$

exists.

Theorem 6.1 is proven for local rings R which are regular, or normal excellent and equicharacteristic in [6].

Remark 6.2. *A reduced analytic local ring R satisfies the hypotheses of Theorem 6.1. In fact, an analytic local ring is excellent by Theorem 102 on page 291 [29] and a reduced, excellent local ring is unramified by (x) of Scholie 7.8.3 [18].*

Proof. We may assume that R is complete, by replacing R, I_i, J_i by $\hat{R}, I_i\hat{R}, J_i\hat{R}$.

First suppose that R is analytically irreducible. Then either $I_i = J_i = 0$ for all $i \geq 1$ or $I_1 \neq 0$ (and $J_1 \neq 0$). We may thus assume that $I_1 \neq 0$. We will prove the theorem in this case. We will apply the method of Theorem 4.2. Construct the regular local ring S by the argument of the proof of Theorem 4.2.

Let ν be the valuation of $Q(R)$ constructed from S in the proof of Theorem 4.2, with associated valuation ideals K_λ in the valuation ring V_ν of ν . Let $k = R/m_R$ and $k' = S/m_S = V_\nu/m_\nu$.

By (14), there exists $\alpha \in \mathbb{Z}_+$ such that

$$K_{\alpha n} \cap R \subset m_R^n$$

for all $n \in \mathbb{Z}_+$. We have that

$$K_{\alpha n} \cap I_n = K_{\alpha n} \cap J_n$$

for all n . Thus

$$(31) \quad \ell_R(J_n/I_n) = \ell_R(J_n/K_{\alpha n} \cap J_n) - \ell_R(I_n/K_{\alpha n} \cap I_n)$$

for all n . Let $\beta = \alpha c$. For $t \geq 1$, let

$$\Gamma(J_*)^{(t)} = \left\{ (n_1, \dots, n_d, i) \mid \dim_k J_i \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d} / J_i \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d}^+ \geq t \right. \\ \left. \text{and } n_1 + \dots + n_d \leq \beta i, \right\}$$

and

$$\Gamma(I_*)^{(t)} = \left\{ (n_1, \dots, n_d, i) \mid \dim_k I_i \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d} / I_i \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d}^+ \geq t \right. \\ \left. \text{and } n_1 + \dots + n_d \leq \beta i \right\}.$$

We have that

$$(32) \quad \ell_R(J_n/I_n) = \left(\sum_{t=1}^{[k':k]} \# \Gamma(J_*)_n^{(t)} \right) - \left(\sum_{t=1}^{[k':k]} \# \Gamma(I_*)_n^{(t)} \right)$$

as explained in the proof of Theorem 4.2. As in the proof of Lemma 4.5 (this is where we need $I_1 \neq 0$ and thus $J_1 \neq 0$), we have that $\Gamma(J_*)^{(t)}$ and $\Gamma(I_*)^{(t)}$ satisfy the conditions (5) and (6) of Theorem 3.2. Thus

$$\lim_{n \rightarrow \infty} \frac{\# \Gamma(J_*)_n^{(t)}}{n^d} = \text{vol}(\Delta(\Gamma(J_*))) \text{ and } \lim_{n \rightarrow \infty} \frac{\# \Gamma(I_*)_n^{(t)}}{n^d} = \text{vol}(\Delta(\Gamma(I_*)))^{(t)}$$

by Theorem 3.2. The theorem, in the case when R is analytically irreducible, now follows from (31).

Now suppose that R is only analytically unramified. We may continue to assume that R is complete. Let P_1, \dots, P_s be the minimal primes of R . Let $R_i = R/P_i$ for $1 \leq i \leq s$. Let $T = \bigoplus_{i=1}^s R_i$ and $\varphi : R \rightarrow T$ be the natural inclusion. By Artin-Rees, there exists a positive integer λ such that

$$\omega_n := \varphi^{-1}(m_R^n T) = R \cap m_R^n T \subset m_R^{n-\lambda}$$

for all $n \geq \lambda$. Thus

$$m_R^n \subset \omega_n \subset m_R^{n-\lambda}$$

for all n . We have that

$$\omega_n = \varphi^{-1}(m_R^n T) = \varphi^{-1}(m_R^n R_1 \bigoplus \dots \bigoplus m_R^n R_s) = (m_R^n + P_1) \cap \dots \cap (m_R^n + P_s).$$

Let $\beta = (\lambda + 1)c$. Now $\omega_{\beta n} \subset m_R^{c(\lambda+1)n-\lambda} \subset m_R^{cn}$ for all $n \geq 1$, so that

$$\omega_{\beta n} \cap I_n = \omega_{\beta n} \cap (m_R^{cn} \cap I_n) = \omega_{\beta n} \cap (m_R^{cn} \cap J_n) = \omega_{\beta n} \cap J_n$$

for all $n \geq 1$, so

$$(33) \quad \ell_R(J_n/I_n) = \ell_R(J_n/\omega_{\beta n} \cap J_n) - \ell_R(I_n/\omega_{\beta n} \cap I_n)$$

for all $n \geq 1$.

Define $L_0^j = R$ for $0 \leq j \leq s$, and for $n > 0$, define $L_n^0 = J_n$ and

$$L_n^j = (m_R^{\beta n} + P_1) \cap \dots \cap (m_R^{\beta n} + P_j) \cap J_n$$

for $1 \leq j \leq s$. For fixed j , with $0 \leq j \leq s$, $\{L_n^j\}$ is a graded family of ideals in R . For $n \geq 1$, we have isomorphisms

$$L_n^j / L_n^{j+1} = L_n^j / (m_R^{\beta n} + P_{j+1}) \cap L_n^j \cong L_n^j R_{j+1} / (L_n^j R_{j+1}) \cap m_{R_{j+1}}^{\beta n}$$

for $0 \leq j \leq s-1$, and

$$L_n^s = \omega_{\beta n} \cap J_n.$$

Thus

$$(34) \quad \ell_R(J_n/\omega_{\beta n} \cap J_n) = \sum_{j=0}^{s-1} \ell_R(L_n^j/L_n^{j+1}) = \sum_{j=0}^{s-1} \ell_{R_{j+1}} \left(L_n^j R_{j+1} / (L_n^j R_{j+1}) \cap m_{R_{j+1}}^{\beta n} \right).$$

For some fixed j with $0 \leq j \leq s-1$, let $\bar{R} = R_{j+1}$, $\bar{J}_n = L_n^j \bar{R}$ and $\bar{I}_n = \bar{J}_n \cap m_{\bar{R}}^{\beta n}$. $\{\bar{I}_n\}$ and $\{\bar{J}_n\}$ are graded families of ideals in \bar{R} and $m_{\bar{R}}^{\beta n} \cap \bar{I}_n = m_{\bar{R}}^{\beta n} \cap \bar{J}_n$ for all n .

We have that $\bar{I}_1 = 0$ implies $\bar{I}_i = 0$ for all $i \geq 1$ and $\bar{J}_1 = 0$ implies $\bar{J}_i = 0$ for all $i \geq 1$ by our initial assumptions. Since $\dim \bar{R} \leq \dim R = d$ and \bar{R} is analytically irreducible, by the first part of the proof we have that

$$\lim_{n \rightarrow \infty} \frac{\ell_{\bar{R}}(\bar{J}_n/\bar{I}_n)}{n^d}$$

exists, and from (34), we have that

$$\lim_{n \rightarrow \infty} \frac{\ell_R(J_n/\omega_{\beta n} \cap J_n)}{n^d}$$

exists. The same argument applied to the graded family of ideals $\{I_n\}$ in R implies that

$$\lim_{n \rightarrow \infty} \frac{\ell_R(I_n/\omega_{\beta n} \cap I_n)}{n^d}$$

exists. Finally, (33) implies that the limit

$$\lim_{n \rightarrow \infty} \frac{\ell_R(J_n/I_n)}{n^d}$$

exists. □

If R is a local ring and I is an ideal in R then the saturation of I is

$$I^{\text{sat}} = I : m_R^\infty = \cup_{k=1}^\infty I : m_R^k.$$

Corollary 6.3. *Suppose that R is an analytically unramified local ring of dimension $d > 0$ and I is an ideal in R . Then the limit*

$$\lim_{i \rightarrow \infty} \frac{\ell_R((I^i)^{\text{sat}}/I^i)}{i^d}$$

exists.

Since $(I^n)^{\text{sat}}/I^n \cong H_{m_R}^0(R/I^n)$, the epsilon multiplicity of Ulrich and Validashti [39]

$$\varepsilon(I) = \limsup \frac{\ell_R(H_{m_R}^0(R/I^n))}{n^d/d!}$$

exists as a limit, under the assumptions of Corollary 6.3.

Corollary 6.3 is proven for more general families of modules when R is a local domain which is essentially of finite type over a perfect field k such that R/m_R is algebraic over k in [5]. The corollary is proven with more restrictions on R in Corollary 6.3 [6]. The limit in corollary 6.3 can be irrational, as shown in [11].

Proof. By Theorem 3.4 [36], there exists $c \in \mathbb{Z}_+$ such that each power I^n of I has an irredundant primary decomposition

$$I^n = q_1(n) \cap \cdots \cap q_s(n)$$

where $q_1(n)$ is m_R -primary and $m_R^{nc} \subset q_1(n)$ for all n . Since $(I^n)^{\text{sat}} = q_2(n) \cap \cdots \cap q_s(n)$, we have that

$$I^n \cap m_R^{nc} = m_R^{nc} \cap q_2(n) \cap \cdots \cap q_s(n) = m_R^{nc} \cap (I^n)^{\text{sat}}$$

for all $n \in \mathbb{Z}_+$. Thus the corollary follows from Theorem 6.1, taking $I_i = I^i$ and $J_i = (I^i)^{\text{sat}}$. \square

A stronger version of the previous corollary is true. The following corollary proves a formula proposed by Herzog, Puthenpurakal and Verma in the introduction to [20]. The formula is proven with more restrictions on R in Corollary 6.4 [6].

Suppose that R is a ring, and I, J are ideals in R . Then the n^{th} symbolic power of I with respect to J is

$$I_n(J) = I^n : J^\infty = \cup_{i=1}^\infty I^n : J^i.$$

Corollary 6.4. *Suppose that R is an analytically unramified local ring of dimension d . Suppose that I and J are ideals in R . Let s be the constant limit dimension of $I_n(J)/I^n$ for $n \gg 0$. Suppose that $s < d$. Then*

$$\lim_{n \rightarrow \infty} \frac{e_{m_R}(I_n(J)/I^n)}{n^{d-s}}$$

exists.

Proof. There exists a positive integer n_0 such that the set of associated primes of R/I^n stabilizes for $n \geq n_0$ by [3]. Let $\{p_1, \dots, p_t\}$ be this set of associated primes. We thus have irredundant primary decompositions for $n \geq n_0$,

$$(35) \quad I^n = q_1(n) \cap \cdots \cap q_t(n),$$

where $q_i(n)$ are p_i -primary.

We further have that

$$(36) \quad I^n : J^\infty = \cap_{J \not\subset p_i} q_i(n).$$

Thus $\dim I_n(J)/I^n$ is constant for $n \geq n_0$. Let s be this limit dimension. The set

$$A = \{p \in \cup_{n \geq n_0} \text{Ass}(I_n(J)/I^n) \mid n \geq n_0 \text{ and } \dim R/p = s\}$$

is a finite set. Moreover, every such prime is in $\text{Ass}(I_n(J)/I^n)$ for all $n \geq n_0$. For $n \geq n_0$, we have by the additivity formula (V-2 [35] or Corollary 4.6.8, page 189 [4]), that

$$e_{m_R}(I_n(J)/I^n) = \sum_p \ell_{R_p}((I_n(J)/I^n)_p) e(m_{R/p})$$

where the sum is over the finite set of primes $p \in \text{Spec}(R)$ such that $\dim R/p = s$. This sum is thus over the finite set A .

Suppose that $p \in A$ and $n \geq n_0$. Then

$$I_p^n = \cap q_i(n)_p$$

where the intersection is over the $q_i(n)$ such that $p_i \subset p$, and

$$I_n(J) = \cap q_i(n)_p$$

where the intersection is over the $q_i(n)$ such that $J \not\subset p_i$ and $p_i \subset p$. Thus there exists an index i_0 such that $p_{i_0} = p$ and

$$I_p^n = q_{i_0}(n)_p \cap I_n(J)_p.$$

By (35),

$$(I_p^n)^{\text{sat}} = I_n(J)_p$$

for $n \geq n_0$. Since R_p is analytically unramified (by [34] or Proposition 9.1.4 [37]) and $\dim R_p \leq d - s$, by Corollary 6.3, the limit

$$\lim_{n \rightarrow \infty} \frac{\ell_R((I_n(J)/I_n)_p)}{n^{d-s}}$$

exists. □

We now establish some Volume = Multiplicity formulas.

Theorem 6.5. *Suppose that R is a d -dimensional analytically unramified local ring and $\{I_i\}$ is a graded family of m_R -primary ideals in R . Then*

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d/d!} = \lim_{p \rightarrow \infty} \frac{e(I_p)}{p^d}$$

exists. Here $e(I_p)$ is the multiplicity

$$e(I_p) = e_{I_p}(R) = \lim_{k \rightarrow \infty} \frac{\ell_R(R/I_p^k)}{k^d/d!}.$$

Theorem 6.5 is proven for valuation ideals associated to an Abhyankar valuation in a regular local ring which is essentially of finite type over a field in [15], for general families of m_R -primary ideals when R is a regular local ring containing a field in [31] and when R is a local domain which is essentially of finite type over an algebraically closed field k with $R/m_R = k$ in Theorem 3.8 [28]. It is proven when R is regular or R is analytically unramified with perfect residue field in Theorem 6.5 [6].

Proof. There exists $c \in \mathbb{Z}_+$ such that $m_R^c \subset I_1$.

We first prove the theorem when R is analytically irreducible, and so satisfies the assumptions of Theorem 4.2. We may assume that R is complete. Let ν be the valuation of $Q(R)$ constructed from S in the proof of Theorem 4.2, with associated valuation ideals K_λ in the valuation ring V_ν of ν . Let $k = R/m_R$ and $k' = S/m_S = V_\nu/m_\nu$.

Apply (14) to find $\alpha \in \mathbb{Z}_+$ such that

$$K_{\alpha n} \cap R \subset m_R^n$$

for all $n \in \mathbb{N}$. We have that

$$K_{\alpha c n} \cap R \subset m_R^{cn} \subset I_n$$

for all n .

For $t \geq 1$, let

$$\Gamma(I_*)^{(t)} = \left\{ (n_1, \dots, n_d, i) \mid \begin{array}{l} \dim_k I_i \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d} / I_i \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d}^+ \geq t \\ \text{and } n_1 + \dots + n_d \leq \alpha c i \end{array} \right\},$$

and

$$\Gamma(R)^{(t)} = \left\{ (n_1, \dots, n_d, i) \mid \begin{array}{l} \dim_k R \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d} / R \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d}^+ \geq t \\ \text{and } n_1 + \dots + n_d \leq \alpha c i \end{array} \right\}.$$

As in the proofs of Lemmas 4.4 and 4.5, $\Gamma(I_*)^{(t)}$ and $\Gamma(R)^{(t)}$ satisfy the conditions (5) and (6) of Theorem 3.2 when they are not contained in $\{0\}$. For fixed $p \in \mathbb{Z}_+$ and $t \geq 1$, let

$$\Gamma(I_*)(p)^{(t)} = \left\{ \begin{array}{l} (n_1, \dots, n_d, kp) \mid \dim_k I_p^k \cap K_{n_1\lambda_1+\dots+n_d\lambda_d} / I_p^k \cap K_{n_1\lambda_1+\dots+n_d\lambda_d}^+ \geq t \\ \text{and } n_1 + \dots + n_d \leq \alpha ckp \end{array} \right\}.$$

We have inclusions of semigroups

$$k * \Gamma(I_*)_p^{(t)} \subset \Gamma(I_*)(p)_{kp}^{(t)} \subset \Gamma(I_*)_{kp}^{(t)}$$

for all p, t and k .

By Theorem 3.4, given $\varepsilon > 0$, there exists p_0 such that $p \geq p_0$ implies

$$\text{vol}(\Delta(\Gamma(I_*)^{(t)})) - \frac{\varepsilon}{[k' : k]} \leq \lim_{k \rightarrow \infty} \frac{\#(k * \Gamma(I_*)_p^{(t)})}{k^d p^d}.$$

Thus

$$\text{vol}(\Delta(\Gamma(I_*)^{(t)})) - \frac{\varepsilon}{[k' : k]} \leq \lim_{k \rightarrow \infty} \frac{\#\Gamma(I_*)(p)_{kp}^{(t)}}{k^d p^d} \leq \text{vol}(\Delta(\Gamma(I_*)^{(t)})).$$

Again by Theorem 3.4, we can choose p_0 sufficiently large that we also have that

$$\text{vol}(\Delta(\Gamma(R)^{(t)})) - \frac{\varepsilon}{[k' : k]} \leq \lim_{k \rightarrow \infty} \frac{\#\Gamma(R)_{kp}^{(t)}}{k^d p^d} \leq \text{vol}(\Delta(\Gamma(R)^{(t)})).$$

Now

$$\ell_R(R/I_p^k) = \left(\sum_{t=1}^{[k' : k]} \#\Gamma(R)_{kp}^{(t)} \right) - \left(\sum_{t=1}^{[k' : k]} \#\Gamma(I_*)(p)_{kp}^{(t)} \right)$$

and

$$\ell_R(R/I_n) = \left(\sum_{t=1}^{[k' : k]} \#\Gamma(R)_n^{(t)} \right) - \left(\sum_{t=1}^{[k' : k]} \#\Gamma(I_*)_n^{(t)} \right).$$

By Theorem 3.2,

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d} = \left(\sum_{t=1}^{[k' : k]} \text{vol}(\Delta(\Gamma(R)^{(t)})) \right) - \left(\sum_{t=1}^{[k' : k]} \text{vol}(\Delta(\Gamma(I_*)^{(t)})) \right).$$

Thus

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d} - \varepsilon \leq \lim_{k \rightarrow \infty} \frac{\ell_R(R/I_p^k)}{k^d p^d} = \frac{e(I_p)}{d! p^d} \leq \lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d} + \varepsilon.$$

Taking the limit as $p \rightarrow \infty$, we obtain the conclusions of the theorem.

Now assume that R is analytically unramified. We may assume that R is complete and reduced since

$$\ell_R(R/I_p^k) = \ell_{\hat{R}}(\hat{R}/I_p^k \hat{R}) \text{ and } e_d(I_p, R) = e_d(I_p \hat{R}, \hat{R})$$

for all p, k .

Suppose that the minimal primes of (the reduced ring) R are $\{q_1, \dots, q_s\}$. Let $R_i = R/q_i$. R_i are complete local domains. We have that

$$\frac{e_d(I_p, R)}{p^d} = \sum_{i=1}^s \frac{e_d(I_p R_i, R_i)}{p^d}$$

by the additivity formula (page V-3 [35] or Corollary 4.6.8, page 189 [4]) or directly from Lemma 4.8. We also have that

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d} = \sum_{i=1}^s \lim_{n \rightarrow \infty} \frac{\ell_R(R_i/I_n R_i)}{n^d}$$

by Lemma 4.8. Since each R_i is analytically irreducible, the limits

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R_i/I_n R_i)}{n^d} = \lim_{p \rightarrow \infty} \frac{e_d(I_p R_i, R_i)}{p^d}$$

exist by the earlier proof of this theorem for analytically irreducible local rings. The conclusions of the theorem now follow. \square

Suppose that R is a Noetherian ring, and $\{I_i\}$ is a graded family of ideals in R . Let

$$s = s(I_*) = \limsup \dim R/I_i.$$

Let $i_0 \in \mathbb{Z}_+$ be the smallest integer such that

$$(37) \quad \dim R/I_i \leq s \text{ for } i \geq i_0.$$

For $i \geq i_0$ and p a prime ideal in R such that $\dim R/p = s$, we have that $(I_i)_p = R_p$ or $(I_i)_p$ is p_p -primary.

s is in general not a limit, as is shown by Example 6.6 [6].

Let

$$T = T(I_*) = \{p \in \text{spec}(R) \mid \dim R/p = s \text{ and there exist arbitrarily large } j \text{ such that } (I_j)_p \neq R_p\}.$$

We recall some lemmas from [6].

Lemma 6.6. (Lemma 6.7 [6]) $T(I_*)$ is a finite set.

Lemma 6.7. (Lemma 6.8 [6]) There exist $c = c(I_*) \in \mathbb{Z}_+$ such that if $j \geq i_0$ and $p \in T(I_*)$, then

$$p^{jc} R_p \subset I_j R_p.$$

Let

$$A(I_*) = \{q \in T(I_*) \mid I_n R_q \text{ is } q_q\text{-primary for } n \geq i_0\}.$$

Lemma 6.8. (Lemma 6.9 [6]) Suppose that $q \in T(I_*) \setminus A(I_*)$. Then there exists $b \in \mathbb{Z}_+$ such that $q_q^b \subset (I_n)_q$ for all $n \geq i_0$.

We obtain the following asymptotic additivity formula. It is proven in Theorem 6.10 [6], with the additional assumption that R is regular or analytically unramified of equicharacteristic zero.

Theorem 6.9. Suppose that R is a d -dimensional analytically unramified local ring and $\{I_i\}$ is a graded family of ideals in R . Let $s = s(I_*) = \limsup \dim R/I_i$. Suppose that $s < d$. Then

$$\lim_{n \rightarrow \infty} \frac{e_s(m_R, R/I_n)}{n^{d-s}/(d-s)!} = \sum_q \left(\lim_{k \rightarrow \infty} \frac{e((I_k)_q)}{k^{d-s}} \right) e(m_{R/q})$$

where the sum is over all prime ideals q such that $\dim R/q = s$.

Proof. Let i_0 be the (smallest) constant satisfying (37). By the additivity formula (V-2 [35] or Corollary 4.6.8, page 189 [4]), for $i \geq i_0$,

$$e_s(m_R, R/I_i) = \sum_p \ell_{R_p}(R_p/(I_i)_p) e_{m_R}(R/p)$$

where the sum is over all prime ideals p of R with $\dim R/p = s$. By Lemma 6.6, for $i \geq i_0$, the sum is actually over the finite set $T(I_*)$ of prime ideals of R .

For $p \in T(I_*)$, R_p is a local ring of dimension $\leq d - s$. Further, R_p is analytically unramified (by [34] or Prop 9.1.4 [37]). By Lemma 6.7, and by Theorem 4.7, replacing $(I_i)_p$ with $p_p^{i_c}$ if $i < i_0$, we have that

$$\lim_{i \rightarrow \infty} \frac{\ell_{R_p}(R_p/(I_i)_p)}{i^{d-s}}$$

exists. Further, this limit is zero if $p \in T(I_*) \setminus A(I_*)$ by Lemma 6.8, and since $s < d$. Finally, we have

$$\lim_{i \rightarrow \infty} \frac{\ell_{R_q}(R_q/(I_i)_q)}{i^{d-s}/(d-s)!} = \lim_{k \rightarrow \infty} \frac{e_{(I_k)_q}(R_q)}{k^{d-s}}$$

for $q \in A(I_*)$ by Theorem 6.5. □

7. KODAIRA-ITAKA DIMENSION ON PROPER k -SCHEMES

Suppose that X is a d -dimensional proper scheme over a field k , and \mathcal{L} is a line bundle on X . Then under the natural inclusion of rings $k \subset \Gamma(X, \mathcal{O}_X)$, we have that the section ring

$$\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n)$$

is a graded k -algebra. Each $\Gamma(X, \mathcal{L}^n)$ is a finite dimensional k -vector space since X is proper over k . In particular, $\Gamma(X, \mathcal{O}_X)$ is an Artin ring. A graded k -subalgebra $L = \bigoplus_{n \geq 0} L_n$ of a section ring of a line bundle \mathcal{L} on X is called a *graded linear series* for \mathcal{L} .

We define the *Kodaira-Itaka dimension* $\kappa = \kappa(L)$ of a graded linear series L as follows. Let

$$\sigma(L) = \max \left\{ m \mid \begin{array}{l} \text{there exists } y_1, \dots, y_m \in L \text{ which are homogeneous of positive} \\ \text{degree and are algebraically independent over } k \end{array} \right\}.$$

$\kappa(L)$ is then defined as

$$\kappa(L) = \begin{cases} \sigma(L) - 1 & \text{if } \sigma(L) > 0 \\ -\infty & \text{if } \sigma(L) = 0 \end{cases}$$

This definition is in agreement with the classical definition for line bundles on normal projective varieties (Definition in Section 10.1 [21] or Chapter 2 [27]).

Lemma 7.1. *Suppose that L is a graded linear series on a d -dimensional proper scheme X over a field k . Then*

$$(38) \quad \kappa(L) \leq d = \dim X.$$

2) *There exists a positive constant γ such that*

$$(39) \quad \dim_k L_n < \gamma n^d$$

for all n .

- 3) Suppose that $\kappa(L) \geq 0$. Then there exists a positive constant α and a positive integer e such that

$$(40) \quad \dim_k L_{en} > \alpha n^{\kappa(L)}$$

for all positive integers n .

- 4) Suppose that X is reduced and L is a graded linear series on X . Then $\kappa(L) = -\infty$ if and only if $L_n = 0$ for all $n > 0$.

We will show in Theorem 10.2 that (39) of Lemma 7.1 can be sharpened to the statement that there exists a positive constant γ such that

$$(41) \quad \dim_k L_n < \gamma n^e$$

where $e = \max\{\kappa(L), \dim \mathcal{N}_X\}$, where \mathcal{N}_X is the nilradical of X (defined in the section on notations and conventions). By Theorem 10.3, (41) is the best bound possible.

To prove Lemma 7.1, we need the following lemma.

Lemma 7.2. *Suppose that L is a graded linear series on a projective scheme X over a field k . Then*

$$\sigma(L) = \text{Krull dimension}(L)$$

and

$$(42) \quad \kappa(L) = \begin{cases} \text{Krull dimension}(L) - 1 & \text{if Krull dimension}(L) > 0 \\ -\infty & \text{if Krull dimension}(L) = 0. \end{cases}$$

Proof. We first prove the lemma with the assumption that L is a finitely generated k -algebra. In the case when $L_0 = k$, the lemma follows from graded Noether normalization (Theorem 1.5.17 [4]). For a general graded linear series L , we always have that $k \subset L_0 \subset \Gamma(X, \mathcal{O}_X)$, which is a finite dimensional k -vector space since X is a projective k -scheme. Let $m = \sigma(L)$ and $y_1, \dots, y_m \in L$ be homogeneous elements of positive degree which are algebraically independent over k . Extend to homogeneous elements of positive degree y_1, \dots, y_n which generate L as an L_0 -algebra. Let $B = k[y_1, \dots, y_n]$. We have that $\sigma(L) \leq \sigma(B) \leq \sigma(L)$ so $\sigma(B) = \sigma(L)$. By the first case ($L_0 = k$) proven above, we have that $\sigma(B) = \text{Krull dimension}(B)$. Since L is finite over B , we have that $\text{Krull dimension}(L) = \text{Krull dimension}(B)$. Thus the lemma holds when L is a finitely generated k -algebra.

Now suppose that L is an arbitrary graded linear series on X . Since X is projective over k , we have an expression $X = \text{Proj}(A)$ where A is the quotient of a standard graded polynomial ring $R = k[x_0, \dots, x_n]$ by a homogeneous ideal I , which we can take to be saturated; that is, (x_0, \dots, x_n) is not an associated prime of I . Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the associated primes of I . By graded prime avoidance (Lemma 1.5.10 [4]) there exists a form F in $k[x_0, \dots, x_n]$ of some positive degree c such that $F \notin \cup_{i=1}^t \mathfrak{p}_i$. Then F is a nonzero divisor on A , so that $A \xrightarrow{F} A(c)$ is 1-1. Sheafifying, we have an injection

$$(43) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(c).$$

Since $\mathcal{O}_X(c)$ is ample on X , there exists $f > 0$ such that $\mathcal{A} := \mathcal{L} \otimes \mathcal{O}_X(cf)$ is ample. From (43) we then have a 1-1 \mathcal{O}_X -module homomorphism $\mathcal{O}_X \rightarrow \mathcal{O}_X(cf)$, and a 1-1 \mathcal{O}_X -module homomorphism $\mathcal{L} \rightarrow \mathcal{A}$, which induces inclusions of graded k -algebras

$$L \subset \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n) \subset B := \bigoplus_{n \geq 0} \Gamma(X, \mathcal{A}^n).$$

There exists a positive integer e such that \mathcal{A}^e is very ample on X . Thus, by Theorem II.5.19 and Exercise II.9.9 [19], $B' = \bigoplus_{n \geq 0} B_{en}$ is finite over a coordinate ring S of X and thus B is a finitely generated k -algebra.

Let L^i be the k -subalgebra generated by L_j for $j \leq i$.

A k -algebra is *subfinite* if it is a subalgebra of a finitely generated k -algebra. We have that

$$L^i \subset L \subset B$$

are subfinite k -algebras. By Corollary 4.7 [23],

$$\text{Krull dimension}(L^i) \leq \text{Krull dimension}(L) \leq \text{Krull dimension}(B) = \dim(X) + 1$$

for all i .

Let

$$P_0 \subset P_1 \subset \cdots \subset P_r$$

be a chain of distinct prime ideals in L with $r = \text{Krull dimension}(L)$. Since $\bigcup_{i=1}^{\infty} L^i = L$, there exists n_0 such that

$$P_0 \cap L^i \subset P_1 \cap L^i \subset \cdots \subset P_r \cap L^i$$

is a chain of distinct prime ideals in L^i for $i \geq n_0$, and so

$$\text{Krull dimension}(L) = \text{Krull dimension}(L^i)$$

for $\geq n_0$. For $i \gg 0$ we also have that $\sigma(L) = \sigma(L^i)$, so

$$\sigma(L) = \text{Krull dimension}(L).$$

□

We now give the proof of Lemma 7.1.

Formula 2) follows from the following formula: Suppose that \mathcal{M} is a coherent sheaf on X . Then there exists a positive constant γ such that

$$(44) \quad \dim_k \Gamma(X, \mathcal{M} \otimes \mathcal{L}^n) < \gamma n^d$$

for all positive n .

We first prove (44) when X is projective. Let $\mathcal{O}_X(1)$ be a very ample line bundle on X . By Proposition 7.4 [19], there exists a finite filtration of \mathcal{M} by coherent sheaves \mathcal{M}^i with quotients $\mathcal{M}^i / \mathcal{M}^{i-1} \cong \mathcal{O}_{Y_i}(n_i)$, where Y_i are closed integral subschemes of X and $n_i \in \mathbb{Z}$. There exists a number $c > 0$ such that $\mathcal{L} \otimes \mathcal{O}_{Y_i}(c)$ is ample for all i . Let $\mathcal{A} = \mathcal{O}_X(n) \otimes \mathcal{L}$, where $n = c + \max\{|n_i|\}$. For all i and positive n , we have

$$\dim_k \Gamma(X, (\mathcal{M}_i / \mathcal{M}_{i-1}) \otimes \mathcal{L}^n) \leq \dim_k \Gamma(Y_i, \mathcal{O}_{Y_i} \otimes \mathcal{A}^n).$$

This last is a polynomial in n of degree equal to $\dim Y_i$ for large n (by Proposition 8.8a [21]). Thus we obtain the formula (44) in the case that X is projective.

Now suppose that X is proper over k . We prove the formula by induction on $\dim \mathcal{M}$. If $\dim \mathcal{M} = 0$, then $\dim_k \Gamma(X, \mathcal{M}) < \infty$, and $\mathcal{M} \otimes \mathcal{L}^n \cong \mathcal{M}$ for all n , so (44) holds. Suppose that $\dim \mathcal{M} = e (\leq d)$ and the formula is true for coherent \mathcal{O}_X -modules whose support has dimension $< e$. Let \mathcal{I} be the sheaf of ideals on X defined for $\eta \in X$ by

$$\mathcal{I}_\eta = \{f \in \mathcal{O}_{X,\eta} \mid f\mathcal{M}_\eta = 0\}.$$

Let $Y = \text{Spec}(\mathcal{O}_X / \mathcal{I})$, a closed subscheme of X . \mathcal{M} is a coherent \mathcal{O}_Y -module, and Y and \mathcal{M} have the same support, so $\dim Y = e$. By Chow's Lemma, there exists a proper

morphism $\varphi : Y' \rightarrow Y$ such that Y' is projective over k and φ is an isomorphism over an open dense subset of Y . Let \mathcal{K} be the kernel of the natural morphism of \mathcal{O}_Y -modules

$$\mathcal{M} \rightarrow \varphi_* \varphi^* \mathcal{M}.$$

$\dim_k \mathcal{K} < e$ since φ is an isomorphism over a dense open subset of Y . Let $\mathcal{L}' = \varphi^*(\mathcal{L} \otimes \mathcal{O}_Y)$. We have inequalities

$$\dim_k \Gamma(X, \mathcal{M} \otimes \mathcal{L}^n) \leq \dim_k \Gamma(X, \mathcal{K} \otimes \mathcal{L}^n) + \dim_k \Gamma(Y', \varphi^*(\mathcal{M}) \otimes (\mathcal{L}')^n)$$

for all $n \geq 0$, so we get the desired upper bound of (44).

Now we establish 1). Let $\varkappa := \varkappa(L)$. Then there exists an inclusion of a weighted polynomial ring $k[x_0, \dots, x_\varkappa]$ into L . Let f be the least common multiple of the degrees of the x_i . Let $Z = \text{Proj}(k[x_0, \dots, x_\varkappa])$. $\mathcal{O}_Z(f)$ is an ample line bundle on the \varkappa -dimensional weighted projective space Z . Thus there exists a polynomial $Q(n)$ of degree \varkappa such that

$$\dim_k k[x_0, \dots, x_n]_{nf} = \dim_k \Gamma(Z, \mathcal{O}_Z(nf)) = Q(n) \text{ for all } n \gg 0.$$

Thus there exists a positive constant α such that

$$\dim_k L_{nf} \geq \alpha n^\varkappa \text{ for } n \gg 0,$$

whence $\varkappa(L) \leq d$ by 2).

We will now establish formula 3). Suppose that $\varkappa(L) \geq 0$. Let L^i be the k -subalgebra of L generated by L_j for $j \leq i$. For i sufficiently large, we have that $\varkappa(L^i) = \varkappa(L)$. For such an i , since L^i is a finitely generated L_0 -algebra, we have that there exists a number e such that the Veronese algebra L^* defined by $L_n^* = (L^i)_{en}$ is generated as a L_0 -algebra in degree 1. Thus, since L_0 is an Artin ring, and L^* has Krull dimension $\varkappa(L) + 1$ by (42), L^* has a Hilbert polynomial $P(t)$ of degree $\varkappa(L)$, satisfying $\ell_{L_0}(L_n^*) = P(n)$ for $n \gg 0$ (Corollary to Theorem 13.2 [30]), where ℓ_{L_0} denotes length of an L_0 module, and thus $\dim_k L_n^* = (\dim_k L_0)P(n)$ for $n \gg 0$. Thus there exists a positive constant α such that $\dim_k L_n^* > \alpha n^{\varkappa(L)}$ for all n , and so

$$\dim_k L_{en} > \alpha n^{\varkappa(L)}$$

for all positive integers n , which is formula (40).

Finally, we will establish the fourth statement of the lemma. Suppose that X is reduced and $0 \neq L_n$ for some $n > 0$. Consider the graded k -algebra homomorphism $\varphi : k[t] \rightarrow L$ defined by $\varphi(t) = z$ where $k[t]$ is graded by giving t the weight n . The kernel of φ is weighted homogeneous, so it is either 0 or (t^s) for some $s > 1$. Thus if φ is not 1-1 then there exists $s > 1$ such that $z^s = 0$ in L_{ns} . We will show that this cannot happen. Since z is a nonzero global section of $\Gamma(X, \mathcal{L}^n)$, there exists $Q \in X$ such that the image of z in \mathcal{L}_Q^n is σf where $f \in \mathcal{O}_{X,Q}$ is nonzero and σ is a local generator of \mathcal{L}_Q^n . The image of z^s in $\mathcal{L}_Q^{sn} = \sigma^s \mathcal{O}_{X,Q}$ is $\sigma^s f^s$. We have that $f^s \neq 0$ since $\mathcal{O}_{X,Q}$ is reduced. Thus $z^s \neq 0$. We thus have that φ is 1-1, so $\varkappa(L) \geq 0$.

8. LIMITS OF GRADED LINEAR SERIES ON PROPER VARIETIES OVER A FIELD

Suppose that L is a graded linear series on a proper variety X over a field k . The *index* $m = m(L)$ of L is defined as the index of groups

$$m = [\mathbb{Z} : G]$$

where G is the subgroup of \mathbb{Z} generated by $\{n \mid L_n \neq 0\}$.

The following theorem has been proven by Okounkov [33] for section rings of ample line bundles, Lazarsfeld and Mustaă [28] for section rings of big line bundles, and for graded

linear series by Kaveh and Khovanskii [22]. All of these proofs require the assumption that k is algebraically closed. We prove the result here for an arbitrary base field k .

Theorem 8.1. *Suppose that X is a d -dimensional proper variety over a field k , and L is a graded linear series on X with Kodaira-Itaka dimension $\kappa = \kappa(L) \geq 0$. Let $m = m(L)$ be the index of L . Then*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{nm}}{n^\kappa}$$

exists.

In particular, from the definition of the index, we have that the limit

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^\kappa}$$

exists, whenever n is constrained to lie in an arithmetic sequence $a + bm$ ($m = m(L)$ and a an arbitrary but fixed constant), as $\dim_k L_n = 0$ if $m \nmid n$.

An example of a big line bundle where the limit in Theorem 8.1 is an irrational number is given in Example 4 of Section 7 [12].

It follows that $\dim_k L_n = 0$ if $m \nmid n$, and if $\kappa(L) \geq 0$, then there exist positive constants $\alpha < \beta$ such that

$$(45) \quad \alpha n^{\kappa(L)} < \dim_k L_{nm} < \beta n^{\kappa(L)}$$

for all sufficiently large positive integers n

The following theorem is proven by Kaveh and Khovanskii [22] when k is an algebraically closed field (Theorem 3.3 [22]). We prove the theorem for an arbitrary field. Theorem 8.2 is a global analog of Theorem 6.5.

Theorem 8.2. *Suppose that X is a d -dimensional proper variety over a field k , and L is a graded linear series on X with Kodaira-Itaka dimension $\kappa = \kappa(L) \geq 0$. Let $m = m(L)$ be the index of L . Let Y_{nm} be the projective subvariety of $\mathbb{P}^{\dim_k L_{nm}}$ that is the closure of the image of the rational map $L_{nm} : X \dashrightarrow \mathbb{P}_k^{\dim_k L_{nm}-1}$. Let $\deg(Y_{nm})$ be the degree of Y_{nm} in $\mathbb{P}_k^{\dim_k L_{nm}-1}$. Then $\dim Y_{nm} = \kappa$ for $n \gg 0$ and*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{nm}}{n^\kappa} = \lim_{n \rightarrow \infty} \frac{\deg(Y_{nm})}{\kappa! n^\kappa}.$$

Letting t be an indeterminate, $\deg(Y_{nm})$ is the multiplicity of the graded k -algebra $k[L_{nm}t]$ (with elements of $L_{nm}t$ having degree 1).

The proof by Kaveh and Khovanskii actually is valid for a variety X over an arbitrary field k , with the additional assumption that there exists a valuation ν of the function field $k(X)$ of X such that the value group Γ_ν of ν is isomorphic to \mathbb{Z}^d and the residue field $V_\nu/m_\nu = k$. The existence of such a valuation is always true if k is algebraically closed. It is however a rather special condition over non closed fields, as is shown by the following proposition.

Proposition 8.3. *Suppose that X is a d -dimensional projective variety over a field k . Then there exists a valuation ν of the function field $k(X)$ of X such that the value group Γ_ν of ν is isomorphic to \mathbb{Z}^d and the residue field $V_\nu/m_\nu = k$ if and only if there exists a birational morphism $X' \rightarrow X$ of projective varieties such that there exists a nonsingular (regular) k -rational point $Q' \in X'$.*

Proof. First suppose there exists a valuation ν of the function field $k(X)$ of X such that the value group Γ_ν of ν is isomorphic to \mathbb{Z}^d as a group and with residue field $V_\nu/m_\nu = k$. Then ν is an “Abhyankar valuation”; that is

$$\text{trdeg}_k k(X) = d = 0 + d = \text{trdeg}_k V_\nu/m_\nu + \text{rational rank } \Gamma_\nu,$$

with $k = V_\nu/m_\nu$, so there exists a local uniformization of ν by [26]. Let Q be the center of ν on X , so that V_ν dominates $\mathcal{O}_{X,Q}$. $\mathcal{O}_{X,Q}$ is a localization of a k -algebra $k[Z]$ where $Z \subset V_\nu$ is a finite set. By Theorem 1.1 [26], there exists a regular local ring R which is essentially of finite type over k with quotient field $k(X)$ such that V_ν dominates R and $Z \subset R$. Since $k[Z] \subset R$ and V_ν dominates $\mathcal{O}_{X,Q}$, we have that R dominates $\mathcal{O}_{X,Q}$. The residue field $R/m_R = k$ since V_ν dominates R . There exists a projective k -variety X'' such that R is the local ring of a closed k -rational point Q' on X'' , and the birational map $X'' \dashrightarrow X$ is a morphism in a neighborhood of Q' . Let X' be the graph of the birational correspondence between X'' and X . Since $X'' \dashrightarrow X$ is a morphism in a neighborhood of Q' , the projection of X' onto X'' is an isomorphism in a neighborhood of Q' . We can thus identify Q' with a nonsingular k -rational point of X' .

Now suppose that there exists a birational morphism $X' \rightarrow X$ of projective varieties such that there exists a nonsingular k -rational point $Q' \in X'$.

Choose a regular system of parameters y_1, \dots, y_d in $R = \mathcal{O}_{X',Q'}$. $R/m_R = k(Q') = k$, so k is a coefficient field of R . We have that $\hat{R} = k[[y_1, \dots, y_d]]$. We define a valuation $\hat{\nu}$ dominating \hat{R} by stipulating that

$$(46) \quad \hat{\nu}(y_i) = e_i \text{ for } 1 \leq i \leq d$$

where $\{e_i\}$ is the standard basis of the totally ordered group $(\mathbb{Z}^d)_{\text{lex}}$, and $\hat{\nu}(c) = 0$ if c is a nonzero element of k .

If $f \in \hat{R}$ and $f = \sum c_{i_1, \dots, i_d} y_1^{i_1} \cdots y_d^{i_d}$ with $c_{i_1, \dots, i_d} \in k$, then

$$\hat{\nu}(f) = \min\{\nu(y_1^{i_1} \cdots y_d^{i_d}) \mid c_{i_1, \dots, i_d} \neq 0\}.$$

We let ν be the valuation of the function field $k(X)$ which is obtained by restricting $\hat{\nu}$. The value group of ν is $(\mathbb{Z}^d)_{\text{lex}}$.

Suppose that h is in $k(X)$ and $\nu(h) = 0$. Write $h = \frac{f}{g}$ where $f, g \in R$ and $\nu(f) = \nu(g)$. Thus in \hat{R} , we have expansions $f = \alpha y_1^{i_1} \cdots y_d^{i_d} + f'$, $g = \beta y_1^{i_1} \cdots y_d^{i_d} + g'$ where α, β are nonzero elements of k , $\nu(y_1^{i_1} \cdots y_d^{i_d}) = \nu(f) = \nu(g)$ and $\nu(f') > \nu(f)$, $\nu(g') > \nu(g)$. Let $\gamma = \frac{\alpha}{\beta}$ in k . Computing $f - \gamma g$ in \hat{R} , we obtain that $\nu(f - \gamma g) > \nu(f)$, and thus the residue of $\frac{f}{g}$ in V_ν/m_ν is equal to the residue of γ , which is in k . By our construction $k \subset V_\nu$. Thus the residue field $V_\nu/m_\nu = k$. \square

We now proceed to prove Theorems 8.1 and 8.2.

By Chow’s Lemma, there exists a proper birational morphism $\varphi : X' \rightarrow X$ which is an isomorphism over a dense open set, such that X' is projective over k . Since X is integral, we have an inclusion $\Gamma(X, \mathcal{L}^n) \subset \Gamma(X, \varphi^* \mathcal{L}^n)$ for all n . Thus L is a graded linear series for $\varphi^* \mathcal{L}$, on the projective variety X' . In this way, we can assume that X is in fact projective over k .

By [41], X has a closed regular point Q (even though there may be no points which are smooth over k if k is not perfect). Let $R = \mathcal{O}_{X,Q}$. R is a d -dimensional regular local ring. Let $k' = k(Q) = R/m_R$.

Choose a regular system of parameters y_1, \dots, y_d in R . By a similar argument to that of the proof of Theorem 4.2, we may define a valuation ν of the function field $k(X)$ of X

dominating R , by stipulating that

$$(47) \quad \nu(y_i) = e_i \text{ for } 1 \leq i \leq d$$

where $\{e_i\}$ is the standard basis of the totally ordered group $\Gamma_\nu = (\mathbb{Z}^d)_{\text{lex}}$, and $\nu(c) = 0$ if c is a unit in R . As in the proof of Theorem 4.2, we have that the residue field of the valuation ring V_ν of ν is $V_\nu/m_\nu = k(Q) = k'$.

L is a graded linear series for some line bundle \mathcal{L} on X . Since X is integral, \mathcal{L} is isomorphic to an invertible sheaf $\mathcal{O}_X(D)$ for some Cartier divisor D on X . We can assume that Q is not contained in the support of D , after possibly replacing D with a Cartier divisor linearly equivalent to D . We have an induced graded k -algebra isomorphism of section rings

$$\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n) \rightarrow \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nD))$$

which takes L to a graded linear series for $\mathcal{O}_X(D)$. Thus we may assume that $\mathcal{L} = \mathcal{O}_X(D)$. For all n , the restriction map followed by inclusion into V_ν ,

$$(48) \quad \Gamma(X, \mathcal{L}^n) \rightarrow \mathcal{L}_Q = \mathcal{O}_{X,Q} \subset V_\nu$$

is a 1-1 k -vector space homomorphism since X is integral, and we have an induced k -algebra homomorphism (sending $t \mapsto 1$).

$$L \rightarrow \mathcal{O}_{X,Q} \subset V_\nu.$$

Given a nonnegative element γ in the value group $\Gamma_\nu = (\mathbb{Z}^d)_{\text{lex}}$ of ν , we have associated valuation ideals I_γ and I_γ^+ in V_ν defined by

$$I_\gamma = \{f \in V_\nu \mid \nu(f) \geq \gamma\}$$

and

$$I_\gamma^+ = \{f \in V_\nu \mid \nu(f) > \gamma\}.$$

Since $V_\nu/m_\nu = k'$, we have that $I_\lambda/I_\lambda^+ \cong k'$ for all nonnegative elements $\lambda \in \Gamma_\nu$, so

$$(49) \quad \dim_k(I_\gamma/I_\gamma^+) = [k' : k] < \infty$$

for all non negative $\gamma \in \Gamma_\nu$. For $1 \leq t$, let

$$S(L)_n^{(t)} = \{\gamma \in \Gamma_\nu \mid \dim_k L_n \cap I_\gamma / L_n \cap I_\gamma^+ \geq t\}.$$

Since every element of L_n has non negative value (as $L_n \subset V_\nu$), we have by (49) and (48) that

$$(50) \quad \dim_k L_n = \sum_{t=1}^{[k':k]} \#(S(L)_n^{(t)})$$

for all n . For $1 \leq t$, let

$$S(L)^{(t)} = \{(\gamma, n) \mid \gamma \in S(L)_n^{(t)}\}.$$

We have inclusions of semigroups $S(L)^{(t')} \subset S(L)^{(t)}$ if $t < t'$.

Lemma 8.4. *Suppose that $t \geq 1$, $0 \neq f \in L_i$, $0 \neq g \in L_j$ and*

$$\dim_k L_i \cap I_{\nu(f)} / L_i \cap I_{\nu(f)}^+ \geq t.$$

Then

$$(51) \quad \dim_k L_{i+j} \cap I_{\nu(fg)} / L_{i+j} \cap I_{\nu(fg)}^+ \geq t.$$

In particular, the $S(L)^{(t)}$ are subsemigroups of the semigroup \mathbb{Z}^{d+1} whenever $S(L)^{(t)} \neq \emptyset$. We have that $m(S(L)^{(t)}) = m(S(L)^{(1)})$ and $q(S(L)^{(t)}) = q(S(L)^{(1)})$ for all t such that $S(L)^{(t)} \not\subset \{0\}$.

Proof. The proof of (51) and that $S(L)^{(t)}$ are sub semigroups is similar to that of Lemma 4.4.

Suppose that $S(L)^{(t)} \not\subset \{0\}$. $S(L)^{(t)} \subset S(L)^{(1)}$, so

$$(52) \quad m(S(L)^{(1)}) \text{ divides } m(S(L)^{(t)})$$

and

$$(53) \quad q(S(L)^{(t)}) \leq q(S(L)^{(1)}).$$

For all $a \gg 0$, $S(L)_{am(S(L)^{(1)})}^{(1)} \neq \emptyset$. In particular, we can take $a \equiv 1 \pmod{m(S(L)^{(t)})}$. There exists $b > 0$ such that $S(L)_{bm(S(L)^{(t)})}^{(t)} \neq \emptyset$. By (51), we have that

$$S(L)_{am(S(L)^{(1)})+bm(S(L)^{(t)})}^{(t)} \neq \emptyset.$$

Thus $m(S(L)^{(1)}) \in \pi(G(S(L)^{(t)}))$, and by (52), $m(S(L)^{(t)}) = m(S(L)^{(1)})$.

Let $q = q(S(L)^{(1)})$. There exists $n_1 > 0$ and $(\gamma_1, n_1), \dots, (\gamma_q, n_1) \in S(L)_{n_1}^{(1)}$ such that if \mathcal{C}_1 is the cone generated by $(\gamma_1, n_1), \dots, (\gamma_q, n_1)$ in \mathbb{R}^{d+1} , then $\dim \mathcal{C}_1 \cap (\mathbb{R}^d \times \{1\}) = q$. There exists $(\tau, n_2) \in S(L)^{(t)}$ with $n_2 > 0$. Thus

$$(\tau + \gamma_1, n_1 + n_2), \dots, (\tau + \gamma_q, n_1 + n_2) \in S(L)_{n_1+n_2}^{(t)}$$

by (51). Let \mathcal{C}_2 be the cone generated by

$$(\tau + \gamma_1, n_1 + n_2), \dots, (\tau + \gamma_q, n_1 + n_2)$$

in \mathbb{R}^{d+1} . Then $\dim \mathcal{C}_2 \cap (\mathbb{R}^d \times \{1\}) = q$, and $q \leq q(S(L)^{(t)})$. Thus, by (53), $q(S(L)^{(t)}) = q(S(L)^{(1)})$. □

We have that $m = m(L)$ is the common value of $m(S(L)^{(t)})$. Let $q(L)$ be the common value of $q(S(L)^{(t)})$ for $S(L)^{(t)} \not\subset \{0\}$.

There exists a very ample Cartier divisor H on X (at the beginning of the proof we reduced to X being projective) such that $\mathcal{O}_X(D) \subset \mathcal{O}_X(H)$ and the point Q of X (from the beginning of the proof) is not contained in the support of H . Let $A_n = \Gamma(X, \mathcal{O}_X(nH))$ and A be the section ring $A = \bigoplus_{n \geq 0} A_n$. After possibly replacing H with a sufficiently high multiple of H , we may assume that A is generated in degree 1 as a $k'' = \Gamma(X, \mathcal{O}_X)$ -algebra. $[k'' : k] < \infty$ since X is projective. The k -algebra homomorphism $L \rightarrow V_\nu$ defined after (48) extends to a k -algebra homomorphism $L \subset A \rightarrow V_\nu$. Let

$$T_n = \{\gamma \in \Gamma_\nu \mid A_n \cap I_\gamma / A_n \cap I_\gamma^+ \neq 0\},$$

and $T = \{(\gamma, n) \mid \gamma \in T_n\}$. T is a subsemigroup of \mathbb{Z}^{d+1} by the argument of Lemma 8.4, and we have inclusions of semigroups $S^{(t)} \subset T$ for all t .

By our construction, A is naturally a graded subalgebra of the graded algebra $\mathcal{O}_{X,Q}[t]$. Since H is ample on X , we have that $A_{(0)} = k(X)$, where $A_{(0)}$ is the set of elements of degree 0 in the localization of A at the set of nonzero homogeneous elements of A . Thus for $1 \leq i \leq d$, there exists $f_i, g_i \in A_{n_i}$, for some n_i , such that $\frac{f_i}{g_i} = y_i$. Thus

$$(e_i, 0) = (\nu(y_i), 0) = (\nu(f_i), n_i) - (\nu(g_i), n_i) \in G(T).$$

for $1 \leq i \leq d$. Since $A_1 \neq 0$, we then have that $(0, 1) \in G(T)$. Thus $G(T) = \mathbb{Z}^{d+1}$, so $L(T) = \mathbb{R}^{d+1}$, $\partial M(T) = \mathbb{R}^d \times \{0\}$ and $q(T) = d$ (with the notation of Section 3 on cones and semigroups). For all $n \gg 0$ we have a bound

$$|T_n| \leq \dim_k A_n = [k'' : k] \dim_{k''} A_n = [k'' : k] P_A(n)$$

where $P_A(n)$ is the Hilbert polynomial of the k'' -algebra A , which has degree $\dim X = d = q(T)$. Thus, by Theorem 1.18 [22], T is a strongly nonnegative semigroup. Since the $S(L)^{(t)}$ are subsemigroups of T , they are also strongly nonnegative, so by Theorem 3.1 and (50), we have that

$$(54) \quad \lim_{n \rightarrow \infty} \frac{\dim_k L_{nm}}{n^{q(L)}} = \sum_{t=1}^{[k' : k]} \lim_{n \rightarrow \infty} \frac{\#(S(L)_{nm}^{(t)})}{n^{q(L)}} = \sum_{t=1}^{[k' : k]} \frac{\text{vol}_{q(L)}(\Delta(S(L)^{(t)}))}{\text{ind}(S(L)^{(t)})}$$

exists.

Let Y_{pm} be the varieties defined in the statement of Theorem 8.2. Let $d(pm) = \dim Y_{pm}$. The coordinate ring of Y_{pm} is the k -subalgebra $L^{[pm]} := k[L_{pm}]$ of L (but with the grading giving elements of L_{pm} degree 1). The Hilbert polynomial $P_{Y_{pm}}(n)$ of Y_{pm} (Section I.7 [19] or Theorem 4.1.3 [4]) has the properties that

$$(55) \quad P_{Y_{pm}}(n) = \frac{\deg(Y_{pm})}{d(pm)!} n^{d(pm)} + \text{lower order terms}$$

and

$$(56) \quad \dim_k L_{npm}^{[pm]} = P_{Y_{pm}}(n)$$

for $n \gg 0$. We have that

$$(57) \quad \lim_{n \rightarrow \infty} \frac{\dim_k (L^{[pm]})_{npm}}{n^{d(pm)}} = \frac{\deg(Y_{pm})}{d(pm)!}.$$

Suppose that t is such that $1 \leq t \leq [k' : k]$ and $S(L)^{(t)} \not\subset \{0\}$. By Lemma 8.4, for p sufficiently large, we have that $m(S(L^{[pm]})^{(t)}) = mp$. Let \mathcal{C} be the closed cone generated by $S(L)_{pm}^{(t)}$ in \mathbb{R}^{d+1} . We also have that

$$\dim(\mathcal{C} \cap (\mathbb{R}^d \times \{1\})) = \dim(\Delta(S(L)^{(t)})) = q(L)$$

for p sufficiently large (the last equality is by Lemma 8.4). Since $S(L)_{pm}^{(t)} = S(L^{[pm]})_{pm}^{(t)}$, we have that

$$\dim(\mathcal{C} \cap (\mathbb{R}^d \times \{1\})) \leq \dim(\Delta(S(L^{[pm]})^{(t)})) \leq \dim(\Delta(L)^{(t)}).$$

Thus

$$(58) \quad q(S(L^{[pm]})^{(t)}) = q(L)$$

for all p sufficiently large.

By the definition of Kodaira-Iitaka dimension, we also have that

$$(59) \quad \kappa(L^{[pm]}) = \kappa(L)$$

for p sufficiently large.

Now by graded Noether normalization (Section I.7 [19] or Theorem 1.5.17[4]), the finitely generated k -algebra $L^{[pm]}$ satisfies

$$(60) \quad d(pm) = \dim Y_{pm} = \text{Krull dimension}(L^{[pm]}) - 1 = \kappa(L^{[pm]}).$$

We have that

$$(61) \quad \frac{1}{[k' : k]} \dim_k L_{n^{pm}}^{[pm]} \leq \#(S(L^{[pm]})_{n^{pm}}^{(1)}) \leq \dim_k L_{n^{pm}}^{[pm]}$$

for all n . $S(L^{[pm]})^{(1)}$ is strongly nonnegative since $S(L^{[pm]})^{(1)} \subset S(L)^{(1)}$ (or since $L^{[pm]}$ is a finitely generated k -algebra). It follows from Theorem 3.1, (61), (56), (55) and (60) that

$$(62) \quad q(S(L^{[pm]})^{(1)}) = d(pm) = \kappa(L^{[pm]}).$$

From (58), (62) and (59), we have that

$$(63) \quad q(L) = \kappa(L) = \kappa.$$

Theorem 8.1 now follows from (54) and (63). We now prove Theorem 8.2. For all p , we have inequalities

$$\sum_{t=1}^{[k':k]} \#(n * S(L)_{mp}^{(t)}) \leq \sum_{t=1}^{[k':k]} \#(S(L^{[mp]})_{nmp}^{(t)}) \leq \sum_{t=1}^{[k':k]} \#(S(L)_{nmp}^{(t)}).$$

The second term in the inequality is $\dim_k(L^{[pm]})_{nmp}$ and the third term is $\dim_k L_{nmp}$. Dividing by $n^\kappa p^\kappa$, and taking the limit as $n \rightarrow \infty$, we obtain from Theorem 3.3, (63) and (54) for the first term and (57), (59) and (60) for the second term, that for given $\varepsilon > 0$, we can take p sufficiently large that

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{nm}}{n^\kappa} - \varepsilon \leq \frac{\deg(Y_{pm})}{\kappa! p^\kappa} \leq \lim_{n \rightarrow \infty} \frac{\dim_k L_{nm}}{n^\kappa}.$$

Taking the limit as p goes to infinity then proves Theorem 8.2.

Theorem 8.5. (*Fujita Approximation*) Suppose that D is a big Cartier divisor on a complete variety X of dimension d over a field k , and $\varepsilon > 0$ is given. Then there exists a projective variety Y with a birational morphism $f : Y \rightarrow X$, a nef and big \mathbb{Q} -divisor N on Y , and an effective \mathbb{Q} -divisor E on Y such that there exists $n \in \mathbb{Z}_{>0}$ so that nD , nN and nE are Cartier divisors with $f^*(nD) \sim nN + nE$, where \sim denotes linear equivalence, and

$$\text{vol}_Y(N) \geq \text{vol}_X(D) - \varepsilon.$$

Proof. By taking a Chow cover by a birational morphism, which is an isomorphism in codimension one, we may assume that X is projective over k . This theorem was proven over an algebraically closed field of characteristic zero by Fujita [16] (c.f. Theorem 10.35 [27]). It is proven in Theorem 3.4 and Remark 3.4 [28] over an arbitrary algebraically closed field (using Okounkov bodies) and by Takagi [38] using de Jong's alterations [13].

We give a proof for an arbitrary field. The conclusions of Theorem 3.3 [28] over an arbitrary field follow from Theorem 8.2 and formula (57), taking the L_n of Theorem 8.2 to be the $H^0(X, \mathcal{O}_X(nD))$ of Theorem 3.3 [28]. $m = 1$ in Theorem 8.2 since D is big. Then the $V_{k,p}$ of Theorem 3.3 [28] are the $L_{kp}^{[p]}$ of the proof of Theorem 8.2.

The proof of Remark 3.4 [28] is valid over an arbitrary field, using the strengthened form of Theorem 3.3 [28] given above, from which the approximation theorem follows. \square

9. LIMITS ON REDUCED PROPER SCHEMES OVER A FIELD

Suppose that X is a proper scheme over a field k and L is a graded linear series for a line bundle \mathcal{L} on X . Suppose that Y is a closed subscheme of X . Set $\mathcal{L}|Y = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. Taking global sections of the natural surjections

$$\mathcal{L}^n \xrightarrow{\varphi_n} (\mathcal{L}|Y)^n \rightarrow 0,$$

for $n \geq 1$ we have induced short exact sequences of k -vector spaces

$$(64) \quad 0 \rightarrow K(L, Y)_n \rightarrow L_n \rightarrow (L|Y)_n \rightarrow 0,$$

where

$$(L|Y)_n := \varphi_n(L_n) \subset \Gamma(Y, (\mathcal{L}|Y)^n)$$

and $K(L, Y)_n$ is the kernel of $\varphi_n|L_n$. Defining $K(L, Y)_0 = k$ and $(L|Y)_0 = \varphi_0(L_0)$, we have that $L|Y = \bigoplus_{n \geq 0} (L|Y)_n$ is a graded linear series for $\mathcal{L}|Y$ and $K(L, Y) = \bigoplus_{n \geq 0} K(L, Y)_n$ is a graded linear series for \mathcal{L} .

Lemma 9.1. *Suppose that X is a reduced proper scheme over a field k and X_1, \dots, X_s are the irreducible components of X . Suppose that L is a graded linear series on X . Then*

$$\kappa(L) = \max\{\kappa(L|X_i) \mid 1 \leq i \leq s\}.$$

Proof. L is a graded linear series for a line bundle \mathcal{L} on X . Let X_1, \dots, X_s be the irreducible components of X . Since X is reduced, we have a natural inclusion

$$0 \rightarrow \mathcal{O}_X \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{X_i}.$$

There is a natural inclusion of k -algebras

$$\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n) \rightarrow \bigoplus_{i=1}^s \left(\bigoplus_{n \geq 0} \Gamma(X_i, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i}) \right),$$

which induces an inclusion of k -algebras

$$(65) \quad L \rightarrow \bigoplus_{i=1}^s L|X_i.$$

Suppose that i is such that $1 \leq i \leq s$. Set $t = \kappa(L|X_i)$. Then by the definition of Kodaira-Iitaka dimension, there exists a graded inclusion of k -algebras $\varphi : k[z_1, \dots, z_t] \rightarrow L|X_i$ where $k[z_1, \dots, z_t]$ is a graded polynomial ring. Since the projection $L \rightarrow L|X_i$ is a surjection, we have a lift of φ to a graded k -algebra homomorphism into L , which is 1-1, so that $\kappa(L) \geq t$. Thus

$$\kappa(L) \geq \max\{\kappa(L|X_i) \mid 1 \leq i \leq s\}.$$

Let $\kappa = \kappa(L)$. Then there exists a 1-1 k -algebra homomorphism $\varphi : k[z_1, \dots, z_q] \rightarrow L$ where $k[z_1, \dots, z_q]$ is a positively graded polynomial ring. Let $\varphi_i : k[z_1, \dots, z_q] \rightarrow L|X_i$ be the induced homomorphisms, for $1 \leq i \leq s$. Let \mathfrak{p}_i be the kernel of φ_i . Since (65) is 1-1, we have that $\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_s = (0)$. Since $k[z_1, \dots, z_q]$ is a domain, this implies that some $\mathfrak{p}_i = (0)$. Thus φ_i is 1-1 and we have that $\kappa(L|X_i) \geq \kappa(L)$. □

Theorem 9.2. *Suppose that X is a reduced proper scheme over a field k . Let L be a graded linear series on X with Kodaira-Iitaka dimension $\kappa = \kappa(L) \geq 0$. Then there exists a positive integer r such that*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{a+nr}}{n^\kappa}$$

exists for any fixed $a \in \mathbb{N}$.

The theorem says that

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^\kappa}$$

exists if n is constrained to lie in an arithmetic sequence $a + br$ with r as above, and for some fixed a . The conclusions of the theorem are a little weaker than the conclusions of Theorem 8.1 for varieties. In particular, the index $m(L)$ has little relevance on reduced but not irreducible schemes (as shown by the example after Theorem 10.2 and Example 9.5).

Proof. Let X_1, \dots, X_s be the irreducible components of X . Define graded linear series M^i on X by $M^0 = L$, $M^i = K(M^{i-1}, X_i)$ for $1 \leq i \leq s$. By (64), for $n \geq 1$, we have exact sequences of k -vector spaces

$$0 \rightarrow (M^{j+1})_n = K(M^j, X_{j+1})_n \rightarrow M_n^j \rightarrow (M^j|X_{j+1})_n \rightarrow 0$$

for $0 \leq j \leq s-1$, and thus

$$M_n^j = \text{Kernel}(L_n \rightarrow \bigoplus_{i=1}^j (L|X_i)_n)$$

for $1 \leq j \leq s$. The natural map $L \rightarrow \bigoplus_{i=1}^s L|X_i$ is an injection of k -algebras since X is reduced. Thus $M_n^s = (0)$, and

$$(66) \quad \dim_k L_n = \sum_{i=1}^s \dim_k (M^{i-1}|X_i)_n$$

for all n . Let $r = \text{LCM}\{m(M^{i-1}|X_i) \mid \kappa(M^{i-1}|X_i) = \kappa(L)\}$. The theorem now follows from Theorem 8.1 applied to each of the X_i with $\kappa(M^{i-1}|X_i) = \kappa(L)$ (we can start with an X_1 with $\kappa(L|X_1) = \kappa(L)$). \square

Corollary 9.3. *Suppose that X is a reduced projective scheme over a perfect field k . Let L be a graded linear series on X with $\kappa(L) \geq 0$. Then there exists a positive constant β such that*

$$(67) \quad \dim_k L_n < \beta n^{\kappa(L)}$$

for all n . Further, there exists a positive constant α and a positive integer m such that

$$(68) \quad \alpha n^{\kappa(L)} < \dim_k L_{mn}$$

for all positive integers n .

Proof. Equation (67) follows from (66), since $\dim_k (M^{i-1}|X_i) \leq \dim_k (L|X_i)$ for all i , and since (45) holds on a variety. Equation (68) is immediate from (40). \square

The following lemma is required for the construction of the next example. It follows from Theorem V.2.17 [19] when $r = 1$. The lemma uses the notation of [19].

Lemma 9.4. *Let k be an algebraically closed field, and write $\mathbb{P}^1 = \mathbb{P}_k^1$. Suppose that $r \geq 0$. Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^r)$ with natural projection $\pi : X \rightarrow \mathbb{P}^1$. Then the complete linear system $|\Gamma(X, \mathcal{O}_X(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(1))|$ is base point free, and the only curve contracted by the induced morphism of X is the curve C which is the section of π defined by the projection of $\mathcal{O}(-1)_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}^r$ onto the first factor.*

Proof. We prove this by induction on r .

First suppose that $r = 0$. Then π is an isomorphism, and $X = C$.

$$\mathcal{O}_X(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(1) \cong \pi_* \mathcal{O}_X(1) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1},$$

from which the statement of the lemma follows.

Now suppose that $r > 0$ and the statement of the lemma is true for $r - 1$. Let V_0 be the \mathbb{P}^1 -subbundle of X corresponding to projection onto the first $r - 1$ factors,

$$(69) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \bigoplus \mathcal{O}_{\mathbb{P}^1}^r \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \bigoplus \mathcal{O}_{\mathbb{P}^1}^{r-1} \rightarrow 0.$$

Apply π_* to the exact sequence

$$0 \rightarrow \mathcal{O}_X(1) \otimes \mathcal{O}_X(-V_0) \rightarrow \mathcal{O}_X(1) \rightarrow \mathcal{O}_{V_0}(1) \rightarrow 0$$

to obtain the exact sequence (69), from which we see that $\mathcal{O}_X(V_0) \cong \mathcal{O}_X(1)$ and $V_0 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{r-1})$ with $\mathcal{O}_X(V_0) \otimes \mathcal{O}_{V_0} \cong \mathcal{O}_{V_0}(1)$. Let F be the fiber over a point in \mathbb{P}^1 by π . We have that $\mathcal{O}_X(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}_X(V_0 + F)$. Apply π_* to

$$0 \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_X(V_0 + F) \rightarrow \mathcal{O}_X(V_0 + F) \otimes \mathcal{O}_{V_0} \rightarrow 0$$

to get

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \mathcal{O}_{\mathbb{P}^1} \bigoplus \mathcal{O}_{\mathbb{P}^1}(1)^r \rightarrow \pi_*(\mathcal{O}_{V_0}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow 0.$$

Now take global sections to obtain that the restriction map

$$\Gamma(X, \mathcal{O}_X(V_0 + F)) \rightarrow \Gamma(V_0, \mathcal{O}_{V_0}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(1))$$

is a surjection. In particular, by the induction statement, V_0 contains no base points of $\Lambda = |\Gamma(X, \mathcal{O}_X(V_0 + F))|$. Since any two fibers F over points of \mathbb{P}^1 are linearly equivalent, Λ is base point free.

Suppose that γ is a curve of X which is not contained in V_0 . If $\pi(\gamma) = \mathbb{P}^1$ then $(\gamma \cdot F) > 0$ and $(\gamma \cdot V_0) \geq 0$ so that γ is not contracted by Λ . If γ is in a fiber of F then $(\gamma \cdot F) = 0$. Let $F \cong \mathbb{P}^r$ be the fiber of π containing γ . Let $h = F \cdot V_0$, a hyperplane section of F . Then $(\gamma \cdot V_0) = (\gamma \cdot h)_F > 0$. Thus γ is not contracted by Λ . By induction on r , we have that C is the only curve on V_0 which is contracted by Λ . We have thus proven the induction statement for r . □

Example 9.5. *Let k be an algebraically closed field. Suppose that s is a positive integer and $a_i \in \mathbb{Z}_+$ are positive integers for $1 \leq i \leq s$. Suppose that $d > 1$. Then there exists a connected reduced projective scheme X over k which is equidimensional of dimension d with a line bundle \mathcal{L} on X and a bounded function $\sigma(n)$ such that*

$$\dim_k \Gamma(X, \mathcal{L}^n) = \lambda(n) \binom{d+n-1}{d-1} + \sigma(n),$$

where $\lambda(n)$ is the periodic function

$$\lambda(n) = |\{i \mid n \equiv 0(a_i)\}|.$$

The Kodaira-Iitaka dimension of L is $\kappa(L) = d - 1$. Let $m' = \text{LCM}\{a_i\}$. The limit

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^{d-1}}$$

exists whenever n is constrained to be in an arithmetic sequence $a + bm'$ (with any fixed a). We have that $\dim_k L_n \neq 0$ for all n if some $a_i = 1$, so the conclusions of Theorem 8.1 do not quite hold in this example.

Proof. Let E be an elliptic curve over k . Let p_0, p_1, \dots, p_s be points on E such that the line bundles $\mathcal{O}_E(p_i - p_0)$ have order a_i . Let $S = E \times_k \mathbb{P}_k^{d-1}$, and define line bundles $\mathcal{L}_i = \mathcal{O}_E(p_i - p_0) \otimes \mathcal{O}_{\mathbb{P}^{d-1}}(1)$ on S . The Segre embedding gives a closed embedding of S in \mathbb{P}^r with $r = 3d - 1$. Let $\pi : X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^r) \rightarrow \mathbb{P}^1$ be the projective bundle, and let C be the section corresponding to the surjection of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^r$ onto the first factor. Let b_1, \dots, b_s be distinct points of \mathbb{P}^1 and let F_i be the fiber by π over b_i . Let S_i be an embedding of S in F_i . We can if necessary make a translation of S_i so that the point $c_i = C \cdot F_i$ lies on S_i , but is not contained in $p_j \times \mathbb{P}^{d-1}$ for any j . We have a line bundle \mathcal{L}' on the (disjoint) union T of the S_i defined by $\mathcal{L}'|_{S_i} = \mathcal{L}_i$.

By Lemma 9.4, there is a morphism $\varphi : X \rightarrow Y$ which only contracts the curve C . φ is actually birational and an isomorphism away from C , but we do not need to verify this, as we can certainly obtain this after replacing φ with the Stein factorization of φ . Let $Z = \varphi(T)$. The birational morphism $T \rightarrow Z$ is an isomorphism away from the points c_i , which are not contained on the support of the divisor defining \mathcal{L}' . Thus $\mathcal{L}'|(T \setminus \varphi(C))$ extends naturally to a line bundle \mathcal{L} on Z .

We have a short exact sequence

$$0 \rightarrow \mathcal{O}_Z \rightarrow \bigoplus_{i=1}^s \mathcal{O}_{S_i} \rightarrow \mathcal{F} \rightarrow 0$$

where \mathcal{F} has finite support. Tensoring this sequence with \mathcal{L}^n and taking global sections, we obtain that

$$0 \leq \sum_{i=1}^s \dim_k \Gamma(S_i, \mathcal{L}_i^n) - \dim_k \Gamma(Z, \mathcal{L}^n) \leq \dim_k \mathcal{F}$$

for all n . Since

$$\Gamma(S_i, \mathcal{L}_i^n) \cong \Gamma(E, \mathcal{O}_E(n(p_i - p_0))) \otimes_k \Gamma(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}(n))$$

by the Kuenneth formula, we obtain the conclusions of the example. \square

10. NECESSARY AND SUFFICIENT CONDITIONS FOR LIMITS TO EXIST ON A PROPER SCHEME OVER A FIELD

The nilradical \mathcal{N}_X of a scheme X is defined in the section on notations and conventions.

Lemma 10.1. *Suppose that X is a proper scheme over a field k and L is a graded linear series on X . Then $\kappa(L) = \kappa(L|_{X_{\text{red}}})$.*

Proof. Let \mathcal{L} be a line bundle associated to X . We have a commutative diagram

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \rightarrow & \bigoplus_{n \geq 0} K_n & \rightarrow & \bigoplus_{n \geq 0} L_n & \rightarrow & \bigoplus_{n \geq 0} (L|_{X_{\text{red}}})_n \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n \otimes \mathcal{N}_X) & \rightarrow & \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n) & \rightarrow & \bigoplus_{n \geq 0} \Gamma(X_{\text{red}}, (\mathcal{L}|_{X_{\text{red}}})^n) \end{array}$$

so that $K_n = L_n \cap \Gamma(X, \mathcal{L}^n \otimes \mathcal{N}_X)$ for all n .

Suppose that $\sigma \in \Gamma(X, \mathcal{L}^m)$. X is Noetherian, so there exists $r_0 = r_0(\sigma)$ such that the closed sets $\text{sup}(\sigma^r) = \text{sup}(\sigma^{r_0})$ for all $r \geq r_0$. Thus

$$\begin{aligned} \sigma &\in \Gamma(X, \mathcal{L}^n \otimes \mathcal{N}_X) \text{ if and only if} \\ \sigma_Q &\text{ is torsion in the } \mathcal{O}_{X,Q}\text{-algebra } \bigoplus_{n \geq 0} \mathcal{L}_Q^n \text{ for all } Q \in X, \text{ if and only if} \\ \sigma_Q^{r_0} &= 0 \text{ in } \bigoplus_{n \geq 0} \mathcal{L}_Q^n \text{ for all } Q \in X, \text{ if and only if} \\ \sigma^{r_0} &= 0 \text{ in } \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n) \text{ since } \mathcal{L} \text{ is a sheaf.} \end{aligned}$$

Thus $\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n \otimes \mathcal{N}_X)$ is the nilradical of $\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n)$ and so K is the nilradical of L .

We have that $\kappa(L|X_{\text{red}}) \leq \kappa(L)$ since any injection of a weighted polynomial ring into $L|X_{\text{red}}$ lifts to a graded injection into L .

If A is a weighted polynomial ring which injects into L , then it intersects K in (0) , so there is an induced graded inclusion of A into $L|X_{\text{red}}$. Thus $\kappa(L|X_{\text{red}}) = \kappa(L)$. \square

Theorem 10.2. *Suppose that X is a proper scheme over a field k . Let \mathcal{N}_X be the nilradical of X . Suppose that L is a graded linear series on X . Then*

- 1) *There exists a positive constant γ such that $\dim_k L_n < \gamma n^e$ where*

$$e = \max\{\kappa(L), \dim \mathcal{N}_X\}.$$

- 2) *Suppose that $\dim \mathcal{N}_X < \kappa(L)$. Then there exists a positive integer r such that*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{a+nr}}{n^{\kappa(L)}}$$

exists for any fixed $a \in \mathbb{N}$.

Proof. Let \mathcal{L} be a line bundle associated to L , so that $L_n \subset \Gamma(X, \mathcal{L}^n)$ for all n . Let K_n be the kernel of the surjection $L_n \rightarrow (L|X_{\text{red}})_n$. From the exact sequence

$$0 \rightarrow \mathcal{N}_X \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X_{\text{red}}} \rightarrow 0,$$

we see that $K_n \subset \Gamma(X, \mathcal{N}_X \otimes \mathcal{L}^n)$ for all n . There exists a constant c such that

$$\dim_k \Gamma(X, \mathcal{N}_X \otimes \mathcal{L}^n) < cn^{\dim \mathcal{N}_X}$$

for all n . By Lemma 10.1 and Theorem 9.2, 1. holds and there exists a positive integer r such that for any a ,

$$\lim_{n \rightarrow \infty} \frac{\dim_k (L|X_{\text{red}})_{a+nr}}{n^{\kappa(L)}}$$

exists. Thus the conclusions of the theorem hold. \square

An example showing that the r of the theorem might have to be strictly larger than the index $m(L)$ is obtained as follows. Let X_1 and X_2 be two general linear subspaces of dimension d in \mathbb{P}^{2d} . They intersect transversally in a rational point Q . Let L^i be the graded linear series on X_i defined by

$$L_n^1 = \begin{cases} \Gamma(X_1, \mathcal{O}_{X_1}(n) \otimes \mathcal{O}_{X_1}(-Q)) & \text{if } 2 \mid n \\ 0 & \text{otherwise} \end{cases}$$

and

$$L_n^2 = \begin{cases} \Gamma(X_2, \mathcal{O}_{X_2}(n) \otimes \mathcal{O}_{X_2}(-Q)) & \text{if } 3 \mid n \\ 0 & \text{otherwise} \end{cases}$$

Here $\mathcal{O}_{X_i}(-Q)$ denotes the ideal sheaf on X_i of the point Q . Let X be the reduced scheme whose support is $X_1 \cup X_2$. From the short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X_1} \bigoplus \mathcal{O}_{X_2} \rightarrow k(Q) \rightarrow 0,$$

we see that there is a graded linear series L on X associated to $\mathcal{O}_X(1)$ such that $L|_{X_i} = L^i$ for $i = 1, 2$, and $\dim_k L_n = \dim_k L_n^1 + \dim_k L_n^2$ for all n . Thus

$$\dim_k L_n = \begin{cases} 2\binom{d+n}{d} - 2 & \text{if } n \equiv 0 \pmod{6} \\ 0 & \text{if } n \equiv 1 \text{ or } 5 \pmod{6} \\ \binom{d+n}{d} - 1 & \text{if } n \equiv 2, 3 \text{ or } 4 \pmod{6}. \end{cases}$$

In Theorem 10.3, we give general conditions under which limits do not always exist.

Theorem 10.3. *Suppose that X is a d -dimensional projective scheme over a field k with $d > 0$. Let $r = \dim \mathcal{N}_X$, where \mathcal{N}_X is the nilradical of X . Suppose that $r \geq 0$. Let $s \in \{-\infty\} \cup \mathbb{N}$ be such that $s \leq r$. Then there exists a graded linear series L on X with $\kappa(L) = s$ such that*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^r}$$

does not exist, even when n is constrained to lie in any arithmetic sequence.

Remark 10.4. *The sequence*

$$\frac{\dim_k L_n}{n^r}$$

in Theorem 10.3 must be bounded by Theorem 10.2.

Proof. Let Y be an irreducible component of the support of \mathcal{N}_X which has maximal dimension r . Let S be a homogeneous coordinate ring of X , which we may assume is saturated, so that the natural graded homomorphism $S \rightarrow \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$ is an inclusion. Let P_Y be the homogeneous prime ideal of Y in S . There exist homogeneous elements $z_0, \dots, z_r \in S$ such that if \bar{z}_i is the image of z_i in S/P_Y , then $\bar{z}_1, \dots, \bar{z}_r$ is a homogeneous system of parameters in S/P_Y (by Lemma 1.5.10 and Proposition 1.5.11 [4]). We can assume that $\deg z_0 = 1$ since some linear form in S is not in P_Y , so it is not a zero divisor S/P_Y . We can take z_0 to be this form (If k is infinite, we can take all of the z_i to have degree 1). $k[\bar{z}_0, \dots, \bar{z}_r]$ is a weighted polynomial ring (by Theorem 1.5.17 [4]), so $A := k[z_0, \dots, z_r] \cong k[\bar{z}_0, \dots, \bar{z}_r]$ is a weighted polynomial ring. Let N be the nilradical of S . The sheafification of N is \mathcal{N}_X . P_Y is a minimal prime of N , so there exists a homogeneous element $x \in N$ such that $\text{ann}_S(x) = P_Y$. $N_{P_Y} \neq 0$ in S_{P_Y} , so $(P_Y)_{P_Y} \neq 0$. Thus $x \in P_Y$, since otherwise $0 = (xP_Y)_{P_Y} = (P_Y)_{P_Y}$. Consider the graded k -subalgebra $B := A[x] = k[z_0, \dots, z_r, x]$ of S . We have that $x^2 = 0$. Also, $\text{ann}_A(x) = \text{ann}_S(x) \cap A = P_Y \cap A = (0)$. Suppose that $ax + b = 0$ with $a, b \in A$. Then $xb = 0$, whence $b = 0$ and thus $a = 0$ also. Hence the only relation on B is $x^2 = 0$. Let $d_i = \deg z_i$, $e = \deg x$. Recall that $d_0 = 1$. Let $f = \text{LCM}\{d_0, \dots, d_r, e\}$.

First assume that $r \geq 1$. For $0 \leq \alpha \leq r$, let $M_t^{(\alpha)}$ be the k -vector space of homogeneous forms of degree t in the weighted variables z_0, \dots, z_α . $\mathcal{O}_{Z_\alpha}(f)$ is an ample line bundle on the weighted projective space $Z_\alpha = \text{Proj}(k[z_0, \dots, z_\alpha])$ (If $U_i = \text{Spec}(k[z_0, \dots, z_\alpha]_{(z_i)})$, then $\mathcal{O}_{Z_\alpha}|_{U_i} = z_i^{\frac{f}{d_i}} \mathcal{O}_{U_i}$).

$$\dim_k M_{nf}^{(\alpha)} = \dim_k \Gamma(Z_\alpha, \mathcal{O}_{Z_\alpha}(nf))$$

is thus the value of a polynomial $Q_\alpha(n)$ in n of degree α for $n \gg 0$. Write

$$Q_\alpha(n) = c_\alpha n^\alpha + \text{lower order terms.}$$

Suppose that $s \geq 0$ (and $r \geq 1$). Let $L_0 = k$. For $n \geq 1$, let

$$L_n = z_0^{nf} M_{nf}^s + x z_0^{(n-\sigma(n))f-e} M_{(n+\sigma(n))f}^r \subset B_{2nf} \subset S_{2nf} \subset \Gamma(X, \mathcal{O}_X(2nf))$$

where $\sigma(n)$ is the function of (22). $L_m L_n \subset L_{m+n}$ since $\sigma(j) \geq \sigma(i)$ if $j \geq i$. $L = \bigoplus_{n \geq 0} L_n$ is a graded linear series with $\varkappa(L) = s$. Since B has $x^2 = 0$ as its only relation, we have that

$$\begin{aligned} \dim_k L_n &= \dim_k M_{nf}^s + \dim_k M_{(n+\sigma(n))f}^r \\ &= Q_s(n) + Q_r(n + \sigma(n)), \end{aligned}$$

so that

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^r} = \lim_{n \rightarrow \infty} \left(c_s n^{s-r} + c_r \left(1 + \left(\frac{\sigma(n)}{n} \right)^r \right) \right)$$

which does not exist, even when n is constrained to lie in any arithmetic sequence, since $\lim_{n \rightarrow \infty} \frac{\sigma(n)}{n}$ has this property (as commented after (25)).

Suppose that $s = -\infty$ (and $r \geq 1$). Then define the graded linear series L by

$$L_n = x z_0^{(n-\sigma(n))f-e} M_{(n+\sigma(n))f}^r.$$

Then $\varkappa(L) = -\infty$. We compute as above that

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_n}{n^r} = \lim_{n \rightarrow \infty} c_r \left(1 + \left(\frac{\sigma(n)}{n} \right)^r \right)$$

does not exist, even when n is constrained to lie in any arithmetic sequence.

Now assume that $r = s = 0$. Since $\dim \mathcal{N}_X = 0$, we have injections for all n ,

$$\Gamma(X, \mathcal{N}_X) \cong \Gamma(X, \mathcal{N}_X \otimes \mathcal{O}_X(n)) \rightarrow \Gamma(X, \mathcal{O}_X(n)).$$

In this case Y is a closed point, so that $\dim_k \Gamma(X, \mathcal{I}_Y \otimes \mathcal{O}_X(en))$ goes to infinity as $n \rightarrow \infty$ (we assume that $d = \dim X > 0$). Thus for $g \gg 0$, there exists $h \in \Gamma(X, \mathcal{I}_Y \otimes \mathcal{O}_X(eg))$ such that $h \notin \Gamma(X, \mathcal{N}_X \otimes \mathcal{O}_X(eg))$, so h is not nilpotent in S . $h \in P_Y$ implies $hx = 0$ in S . Define $L_0 = k$ and for $n > 0$,

$$L_n = \begin{cases} kh^n & \text{if } \tau(n) = 0 \\ kh^n + kxz_0^{ng-e} & \text{if } \tau(n) = 1, \end{cases}$$

where $\tau(n)$ is the function of (29). $\tau(n)$ has the property that $\tau(n)$ is not eventually constant, even when n is constrained to lie in an arithmetic sequence.

$L = \bigoplus_{n \geq 0} L_n$ is a graded linear series on X with $\varkappa(L) = 0$ such that $\lim_{n \rightarrow \infty} \dim_k L_n$ does not exist, even when n is constrained to lie in any arithmetic sequence.

The last case is when $r = 0$ and $s = -\infty$. Define $L_0 = k$ and for $n > 0$,

$$L_n = \begin{cases} 0 & \text{if } \tau(n) = 0 \\ kxz_0^{ng-e} & \text{if } \tau(n) = 1. \end{cases}$$

Then $L = \bigoplus_{n \geq 0} L_n$ is a graded linear series on X with $\varkappa(L) = -\infty$ such that $\lim_{n \rightarrow \infty} \dim_k L_n$ does not exist, even when n is constrained to lie in any arithmetic sequence. □

Theorem 10.5. *Suppose that X is a projective nonreduced scheme over a field k . Suppose that $s \in \mathbb{N} \cup \{-\infty\}$ satisfies $s \leq \dim \mathcal{N}_X$. Then there exists a graded linear series L on X with $\varkappa(L) = s$ and a constant $\alpha > 0$ such that*

$$\alpha n^{\dim \mathcal{N}_X} < \dim_k L_{nm}$$

for all $n \gg 0$.

Proof. Let $r = \dim \mathcal{N}_X$. When $r \geq 1$ and $s \leq r$, this is established in the construction of Theorem 10.3. When $r = 0$ and $s = 0$, the graded linear series $L = k[t]$ (with associated line bundle \mathcal{O}_X) has $\varkappa(L) = 0$ and $\dim_k L_n = 1$ for all n , so satisfies the bound.

Suppose that $r = 0$ and $s = -\infty$. Then $0 \neq \Gamma(X, \mathcal{N}_X)$ since the support of \mathcal{N}_X is zero dimensional. Define $L_0 = k$ and $L_n = \Gamma(X, \mathcal{N}_X)$ for $n > 0$. Then $L = \bigoplus_{n \geq 0} L_n$ is a graded linear series for \mathcal{O}_X with $\varkappa(L) = -\infty$ which satisfies the bound. \square

Theorem 10.6. *Suppose that X is a d -dimensional projective scheme over a field k with $d > 0$. Let \mathcal{N}_X be the nilradical of X . Let $\alpha \in \mathbb{N}$. Then the following are equivalent:*

- 1) *For every graded linear series L on X with $\alpha \leq \varkappa(L)$, there exists a positive integer r such that*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{a+nr}}{n^{\varkappa(L)}}$$

exists for every positive integer a .

- 2) *For every graded linear series L on X with $\alpha \leq \varkappa(L)$, there exists an arithmetic sequence $a + nr$ (for fixed r and a depending on L) such that*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{a+nr}}{n^{\varkappa(L)}}$$

exists.

- 3) *The nilradical \mathcal{N}_X of X satisfies $\dim \mathcal{N}_X < \alpha$.*

Proof. 1) implies 2) is immediate. 2) implies 3) follows from Theorem 10.3. 3) implies 1) follows from Theorem 10.2. \square

Theorem 10.7. *Suppose that X is a proper scheme of dimension d over a field k , such that $\dim \mathcal{N}_X < d$ and \mathcal{L} is a line bundle on X . Then the limit*

$$\lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(X, \mathcal{L}^n)}{n^d}$$

exists.

Proof. We first prove the theorem in the case when X is integral (a variety). We may assume that the section ring L of \mathcal{L} has maximal Kodaira-Iitaka dimension d , because the limit is zero otherwise. There then exists a positive constant α and a positive integer e such that

$$\dim_k \Gamma(X, \mathcal{L}^{ne}) > \alpha n^d$$

for all positive integers n by (40). Let H be a hyperplane section of X , giving a short exact sequence

$$0 \rightarrow \mathcal{O}_X(-H) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_H \rightarrow 0.$$

Tensoring with \mathcal{L}^n and taking global sections, we see that $\Gamma(X, \mathcal{L}^{ne} \otimes \mathcal{O}_X(-H)) \neq 0$ for $n \gg 0$ as $q(\mathcal{L}^e \otimes \mathcal{O}_H) \leq \dim(H) = d - 1$. Since H is ample, there exists a positive integer f such that $\mathcal{L} \otimes \mathcal{O}_X(fH)$ is generated by global sections. Thus

$$\Gamma(X, \mathcal{L}^{nef+1}) \cong \Gamma(X, (\mathcal{L}^{nef} \otimes \mathcal{O}_X(-fH)) \otimes (\mathcal{L} \otimes \mathcal{O}_X(fH))) \neq 0$$

for $n \gg 0$. Thus $m(L) = 1$. The theorem in the case when X is a variety thus follows from Theorem 8.1.

Now assume that X is reduced. Let X_1, \dots, X_s be the irreducible components of X . Since X is reduced, we have a natural short exact sequence of \mathcal{O}_X -modules

$$0 \rightarrow \mathcal{O}_X \rightarrow \bigoplus_{n \geq 0} \mathcal{O}_{X_i} \rightarrow \mathcal{F} \rightarrow 0$$

where \mathcal{F} has support of dimension $\leq d - 1$. Tensoring with \mathcal{L}^n , we obtain that

$$\lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(X, \mathcal{L}^n)}{n^d} = \sum_{i=1}^s \lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(X_i, \mathcal{L}^n \otimes \mathcal{O}_{X_i})}{n^d}$$

exists, as $\dim_k \Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$ grows at most like n^{d-1} .

Let $\overline{X} = X_{\text{red}}$ so that $\mathcal{O}_{\overline{X}} = \mathcal{O}_X / \mathcal{N}_X$. From the exact sequence

$$0 \rightarrow \mathcal{N}_X \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\overline{X}} \rightarrow 0$$

and since the support of \mathcal{N}_X has dimension less than d , we have that

$$\lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(X, \mathcal{L}^n)}{n^d} = \lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(\overline{X}, \mathcal{L}^n \otimes \mathcal{O}_{\overline{X}})}{n^d}$$

exists. □

11. NONREDUCED ZERO DIMENSIONAL SCHEMES

The case when $d = \dim X = 0$ is rather special. In fact, the implication 2) implies) of Theorem 10.6 does not hold if $d = 0$, as follows from Proposition 11.1 below. There is however a very precise statement about what does happen in zero dimensional schemes, as we show below.

Proposition 11.1. *Suppose that X is a 0-dimensional irreducible but nonreduced k -scheme and L is a graded linear series on X with $\kappa(L) = 0$. Then there exists a positive integer r such that*

$$\lim_{n \rightarrow \infty} \dim_k L_{a+nr}$$

exists for every positive integer a .

Proof. With our assumptions, $X = \text{Spec}(A)$ where A is a nonreduced Artin local ring, with $\dim_k L < \infty$, and L is a graded k -subalgebra of $\Gamma(X, \mathcal{O}_X)[t] = A[t]$. The condition $\kappa(L) = 0$ is equivalent to the statement that there exists $r > 0$ such that L_r contains a unit u of A . We then have that

$$\dim_k L_{m+r} \geq \dim_k L_m L_r \geq \dim_k L_m$$

for all m . Thus for fixed a , $\dim_k L_{a+nr}$ must stabilize for large n . □

We do not have such good behavior for graded linear series L with $\kappa(L) = \infty$.

Proposition 11.2. *Suppose that X is a 0-dimensional nonreduced k -scheme. Then there exists a graded linear series L on X with $\kappa(L) = -\infty$, such that*

$$\lim_{n \rightarrow \infty} \dim_k L_n$$

does not exist, even when n is constrained to lie in any arithmetic sequence.

Proof. $X = \text{Spec}(A)$ where $A = \bigoplus_{i=1}^s A_i$, with s the number of irreducible components of X and the A_i are Artin local rings with $\dim_k A < \infty$. Let m_{A_1} be the maximal ideal of A_1 . There exists a number $0 < t$ such that $m_{A_1}^t \neq 0$ but $m_{A_1}^{t+1} = 0$. Let $\tau(n)$ be the function defined in (29).

Define a graded linear series L^1 on $\text{Spec}(A_1)$ by $L_n^1 = m_{A_1}^{t+\tau(n)}$. Then $\lim_{n \rightarrow \infty} \dim_k L_n^1$ does not exist, even when n is constrained to lie in an arithmetic sequence. Extend L to a graded linear series L on X with $\varkappa(L) = -\infty$ by setting $L_n = L_n^1 \oplus (0) \oplus \cdots \oplus (0)$. \square

It follows that the conclusions of Proposition 11.1 do not hold in nonreduced 0-dimensional schemes which are not irreducible.

Proposition 11.3. *Suppose that X is a 0-dimensional nonirreducible and nonreduced k -scheme. Then there exists a graded linear series L on X with $\varkappa(L) = 0$, such that*

$$(70) \quad \lim_{n \rightarrow \infty} \dim_k L_n$$

does not exist, even when n is constrained in any arithmetic sequence.

Proof. $X = \text{Spec}(A)$ where $A = A_1 \oplus A_2$, with A_1 an Artin local ring and A_2 an Artin ring. A graded linear series L on X is a graded k -subalgebra of $A[t]$. Let L^2 be a graded linear series on $\text{Spec}(A_2)$ with $\varkappa(L^2) = -\infty$, such that the conclusions of Proposition 11.2 hold. Then the linear series L on X defined by

$$L_n = A_1 \bigoplus (L^2)_n$$

has $\varkappa(L) = 0$, but

$$\lim_{n \rightarrow \infty} \dim_k L_n = 1 + \lim_{n \rightarrow \infty} \dim_k L_n^2$$

does not exist, even when n is constrained to lie in any arithmetic sequence. \square

In particular, the conclusions of Theorem 10.6 are true for 0-dimensional projective k -schemes which are not irreducible.

12. EXAMPLES WITH KODAIRA-IITAKA DIMENSION $-\infty$

It is much easier to construct perverse examples with Kodaira-Iitaka dimension $-\infty$, since the condition $L_m L_n \subset L_{m+n}$ can be trivial in this case. If X is a reduced variety, and L is a graded linear series on X , then it follows from Corollary 9.3 that there is an upper bound $\dim_k L_n < \beta n^{q(L)}$ for all n . However, for nonreduced varieties of dimension d , we only have the upper bound $\dim_k L_n < \gamma n^d$ of (39). Here is an example with $\varkappa(L) = -\infty$ and maximal growth of order n^d .

Example 12.1. *Let k be a field, and let X be the one dimensional projective non reduced k -scheme consisting of a double line in \mathbb{P}_k^2 . Let T be a subset of the positive integers. There exists a graded linear series L for $\mathcal{O}_X(2)$ such that*

$$\dim_k L_n = \begin{cases} n+1 & \text{if } n \in T \\ 0 & \text{if } n \notin T \end{cases}$$

In the example, we have that $\varkappa(L) = -\infty$, but $\dim_k L_n$ is $O(n) = O(n^{\dim X})$.

Proof. We can choose homogeneous coordinates on \mathbb{P}_k^2 so that $X = \text{Proj}(S)$, where $S = k[x_0, x_1, x_2]/(x_1^2)$. Let \bar{x}_i be the classes of x_i in S , so that $S = k[\bar{x}_0, \bar{x}_1, \bar{x}_2]$. Define a graded linear series L for $\mathcal{O}_X(2)$ by defining L_n to be the k -subspace of $\Gamma(X, \mathcal{O}_X(2n))$ spanned by $\{\bar{x}_1 \bar{x}_0^i \bar{x}_2^j \mid i + j = n\}$ if $n \in T$ and $L_n = 0$ if $n \notin T$. Then

$$\dim_k L_n = \begin{cases} n + 1 & \text{if } n \in T \\ 0 & \text{if } n \notin T \end{cases}$$

□

We modify the above example a little bit to find another example with interesting growth.

Theorem 12.2. *Let k be a field, and let X be the one dimensional projective non reduced k -scheme consisting of a double line in \mathbb{P}_k^2 . Let T be any infinite subset of the positive integers \mathbb{Z}_+ such that $\mathbb{Z}_+ \setminus T$ is also infinite. There exists a graded linear series L for $\mathcal{O}_X(2)$ such that*

$$\dim_k L_n = \begin{cases} \lceil \log(n) \rceil & \text{if } n \in T \\ \lceil \frac{\log(n)}{2} \rceil & \text{if } n \notin T \end{cases}$$

In this example we have $\kappa(L) = -\infty$.

Proof. We can choose homogeneous coordinates on \mathbb{P}_k^2 so that $X = \text{Proj}(S)$, where $S = k[x_0, x_1, x_2]/(x_1^2)$. Let \bar{x}_i be the classes of x_i in S , so that $S = k[\bar{x}_0, \bar{x}_1, \bar{x}_2]$. Define

$$\lambda(n) = \begin{cases} \lceil \log(n) \rceil & \text{if } n \in T \\ \lceil \frac{\log(n)}{2} \rceil & \text{if } n \in \mathbb{Z}_+ \setminus T. \end{cases}$$

Define a graded linear series L for $\mathcal{O}_X(1)$ by defining L_n to be the k -subspace of $\Gamma(X, \mathcal{O}_X(n))$ spanned by

$$\bar{x}_0^{n-1} \bar{x}_1, \bar{x}_0^{n-2} \bar{x}_1 \bar{x}_2, \dots, \bar{x}_0^{n-\lambda(n)} \bar{x}_1 \bar{x}_2^{\lambda(n)-1}.$$

Then L_n has the desired property. □

The following is an example of a line bundle on a non reduced scheme for which there is interesting growth. The characteristic $p > 0$ plays a role in the construction.

Example 12.3. *Suppose that $d \geq 1$. There exists an irreducible but nonreduced projective variety Z of dimension d over a field of positive characteristic p , and a line bundle \mathcal{N} on Z , whose Kodaira-Iitaka dimension is $-\infty$, such that*

$$\dim_k \Gamma(Z, \mathcal{N}^n) = \begin{cases} \binom{d+n-1}{d-1} & \text{if } n \text{ is a power of } p \\ 0 & \text{otherwise} \end{cases}$$

In particular, given a positive integer r , there exists at least one integer a with $0 \leq a < r$ such that the limit

$$\lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(Z, \mathcal{N}^n)}{n^{d-1}}$$

does not exist when n is constrained to lie in the arithmetic sequence $a + br$.

Proof. Suppose that p is a prime number such that $p \equiv 2 \pmod{3}$. In Section 6 of [12], a projective genus 2 curve C over an algebraic function field k of characteristic p is constructed, which has a k -rational point Q and a degree zero line bundle \mathcal{L} with the properties that

$$\dim_k \Gamma(C, \mathcal{L}^n \otimes \mathcal{O}_C(Q)) = \begin{cases} 1 & \text{if } n \text{ is a power of } p \\ 0 & \text{otherwise} \end{cases}$$

and

$$(71) \quad \Gamma(C, \mathcal{L}^n) = 0 \text{ for all } n.$$

Let $\mathcal{E} = \mathcal{O}_C(Q) \oplus \mathcal{O}_C$. Let $S = \mathbb{P}(\mathcal{E})$ with natural projection $\pi : S \rightarrow C$, a ruled surface over C . Let C_0 be the section of π corresponding to the surjection onto the second factor $\mathcal{E} \rightarrow \mathcal{O}_C \rightarrow 0$. By Proposition V.2.6 [19], we have that $\mathcal{O}_S(-C_0) \otimes_{\mathcal{O}_S} \mathcal{O}_{C_0} \cong \mathcal{O}_C(Q)$. Let X be the nonreduced subscheme $2C_0$ of S . We have a short exact sequence

$$0 \rightarrow \mathcal{O}_C(Q) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0.$$

Let $\mathcal{M} = \pi^*(\mathcal{L}) \otimes_{\mathcal{O}_S} \mathcal{O}_X$. Then we have short exact sequences

$$(72) \quad 0 \rightarrow \mathcal{L}^n \otimes_{\mathcal{O}_C} \mathcal{O}_C(Q) \rightarrow \mathcal{M}^n \rightarrow \mathcal{L}^n \rightarrow 0.$$

By (72) and (71), we have that

$$\begin{aligned} \dim_k \Gamma(X, \mathcal{M}^n) &= \dim_k \Gamma(C, \mathcal{L}^n \otimes \mathcal{O}_C(Q)) \\ &= \begin{cases} 1 & \text{if } n \text{ is a power of } p \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Now let $Z = X \times \mathbb{P}_k^{d-1}$ and $\mathcal{N} = \mathcal{M} \otimes \mathcal{O}_{\mathbb{P}}(1)$. By the Kuenneth formula, we have that

$$\Gamma(Z, \mathcal{N}^n) = \Gamma(X, \mathcal{M}^n) \otimes_k \Gamma(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}}(n))$$

from which the conclusions of the example follow. \square

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